MATH 153 Practice Final Exam Solutions

1. To determine derivatives:

(a) $f(x) = \frac{4}{5}x^{10} - \dfrac{1}{3x^3} - \dfrac{x}{4} + 8 = \frac{4}{5}x^{10} - \frac{1}{3}x^{-3} - \frac{1}{4}x + 8$

$$f'(x) = \tfrac{4}{5}(10)x^9 - \tfrac{1}{3}(-3)x^{-4} - \tfrac{1}{4} = 8x^9 + \frac{1}{x^4} - \tfrac{1}{4}$$

(b) $g(x) = 3\cos x + 5\csc x - 5\cot x + 12\tan x - \ln x$

$$g'(x) = 3(-\sin x) + 5(-\csc x \cot x) + 12\sec^2 x - \frac{1}{x} = -3\sin x - 5\csc x\cot x + 12\sec^2 x - \frac{1}{x}$$

(c) $h(x) = 7x\cos x - 9e^x \sin x$

$$h'(x) = (7)(\cos x) + (-\sin x)7x - (9e^x)(\sin x) + (\cos x)(-9e^x) = 7\cos x - 7x\sin x - 9e^x\sin x - 9e^x\cos x$$

(d) $u(x) = \dfrac{3x}{4x-5}$

$$u'(x) = \frac{(3)(4x-5) - (4)(3x)}{(4x-5)^2} = \frac{12x - 15 - 12x}{(4x-5)^2} = \frac{-15}{(4x-5)^2}$$

(e) $v(x) = \dfrac{\sin x}{2 + \cos x}$

$$v'(x) = \frac{(\cos x)(2+\cos x) - (-\sin x)(\sin x)}{(2+\cos x)^2} = \frac{2\cos x + \cos^2 x + \sin^2 x}{(2+\cos x)^2} = \frac{2\cos x + +1}{(2+\cos x)^2}$$

(f) $w(x) = 5x\left(\sqrt{x^2+4}\right) - \tan 5x = 5x\left(x^2+4\right)^{\frac{1}{2}} - \tan 5x$

$$w'(x) = (5)\left(\sqrt{x^2+4}\right) + \tfrac{1}{2}\left(x^2+4\right)^{-\frac{11}{2}}(2x)(5x) - \left(\sec^2 5x\right)(5) = 5\left(\sqrt{x^2+4}\right) + \frac{5x^2}{\sqrt{x^2+4}} - 5\sec^2 5x$$

(g) $f(x) = \sec^2(4x) - 2\cot(4x) + 15 = (\sec 4x)^2 - 2\cot(4x) + 15$

$$f'(x) = 2(\sec 4x)(\sec 4x \tan 4x)(4) - 2\left(-\csc^2 4x\right)(4) = 8\sec^2 4x\tan 4x + 8\csc^2 4x$$

(h) $v(x) = 3 - \arcsin\left(\frac{1}{3}x\right) + \arccos\left(\frac{1}{2}x\right)$

$$v'(x) = -\frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\cdot\left(\tfrac{1}{3}\right) - \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\left(\tfrac{1}{2}\right) = -\frac{1}{3\left(\sqrt{1-\frac{x^2}{9}}\right)} - \frac{1}{2\left(\sqrt{1-\frac{x^2}{4}}\right)} = -\frac{1}{\sqrt{9-x^2}} - \frac{1}{\sqrt{4-x^2}}$$

(i) $h(x) = 1 - 4\left(\cos\frac{1}{2}x\right)\left(e^{2x}\right)\ln\left(x^2+1\right)$

$$\begin{aligned} h'(x) &= -4\left[\left(-\sin\tfrac{1}{2}x\right)\left(\tfrac{1}{2}\right)\left(e^{2x}\right)\ln\left(x^2+1\right) + \left(e^{2x}\right)(2)\left(\cos\tfrac{1}{2}x\right)\ln\left(x^2+1\right) + \frac{1}{x^2+1}(2x)\left(\cos\tfrac{1}{2}x\right)\left(e^{2x}\right)\right] \\ &= 2e^{2x}\sin\tfrac{1}{2}x\ln\left(x^2+1\right) - 8e^{2x}\left(\cos\tfrac{1}{2}x\right)\ln\left(x^2+1\right) + \frac{2xe^{2x}\cos\frac{1}{2}x}{x^2+1} \end{aligned}$$

2. Given $f(x) = e^{3x}$:

(a) $f'(x) = \left(e^{3x}\right)(3) = 3e^{3x}$

(b) $f''(x) = 3\left(e^{3x}\right)(3) = 9e^{3x}$
(c) $f'''(x) = 9\left(e^{3x}\right)(3) = 27e^{3x}$
(d) The pattern suggests that $f^{(10)}(x) = 3^{10}e^{3x}$
(e) And $f^{(n)}(x) = 3^n e^{3x}$

3. To find the derivative of the function $g(x) = (2x)^{x^2}$: We write $g(x)$ as y to get $y = (2x)^{x^2}$. Taking logarithms to base e gives

$$\ln y = \ln\left((2x)^{x^2}\right) = x^2 \ln(2x) \qquad \text{I.e.} \qquad \ln y = x^2 \ln(2x)$$

We now take derivatives implicitly to get $\dfrac{1}{y}\dfrac{dy}{dx} = 2x\ln(2x) + \dfrac{1}{2x}(2)\left(x^2\right) = 2x\ln(2x) + x$. Therefore

$$g'(x) = (2x\ln(2x) + x)\,y = (2x\ln(2x) + x)\,(2x)^{x^2}$$

4. To determine $\lim\limits_{x\to 2}\dfrac{x^2-5x+6}{x^2-4}$.

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4} = \lim_{x\to 2}\frac{(x-2)(x-3)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{(x-3)}{(x+2)} = \frac{(2-3)}{(2+4)} = -\frac{1}{6}$$

5. To calculate the derivative of $g(x) = \arctan\left(\dfrac{1}{x^2}\right)$ then show that it simplifies to $g'(x) = \dfrac{-2x}{x^4+1}$:

$$g'(x) = \frac{1}{1+\left(\frac{1}{x^2}\right)^2}\left(-2x^{-3}\right) = \frac{1}{1+\frac{1}{x^4}}\left(-2x^{-3}\right) = \frac{1}{\frac{x^4+1}{x^4}}\left(\frac{-2}{x^3}\right) = \frac{x^4}{x^4+1}\left(\frac{-2}{x^3}\right) = \frac{-2x}{x^4+1}$$

6. A function y is defined implicitly by $x^2 + xy - y^2 = -1$

(a) Taking derivatives implicitly gives

$$2x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0 \qquad \text{which may be re-arranged as} \quad 2x + y = (x+2y)\frac{dy}{dx}$$

Therefore $\dfrac{dy}{dx} = \dfrac{2x+y}{x+2y}$

(b) The slope of the tangent to the graph of y at $(1,-1)$ is the value of $\dfrac{dy}{dx}$ when $x = 1$ and$y = -1$ and it is $\dfrac{2(1)+(-1)}{1+2(-1)} = -1$

(c) The equation of the tangent to the graph of y at $(1,-1)$ given by

$$\frac{y-(-1)}{x-1} = -1 \qquad \text{This may be rearranged as } y+1 = -x+1 \quad \text{OR} \quad y = -x$$

7. The volume V of a sphere is increasing at the rate of 12 cubic centimeters per second. At what rate is its radius r changing when it, (i.e. the radius r), is 8 centimeters?

We are given that $V = \frac{4}{3}\pi r^3$ and that both variables V and r change with time t. Taking derivatives implicitly in the equation $V = \frac{4}{3}\pi r^3$, with respect to time t gives

$$\frac{dV}{dt} = \tfrac{4}{3}\pi\left(3r^2\right)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt} \qquad \text{which implies that} \quad \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{12}{4\pi r^2}$$

because the volume is increasing at the rate of 12 cubic centimeters per second. When $r = 8$, $\dfrac{dr}{dt} = \dfrac{12}{4\pi\left(8^2\right)} = \dfrac{3}{64\pi}$. The radius is increasing at the rate of $\dfrac{3}{64\pi}$ per second.

8. To find the two critical points of $f(x) = 3x^2 - x^3$ and establish their nature.

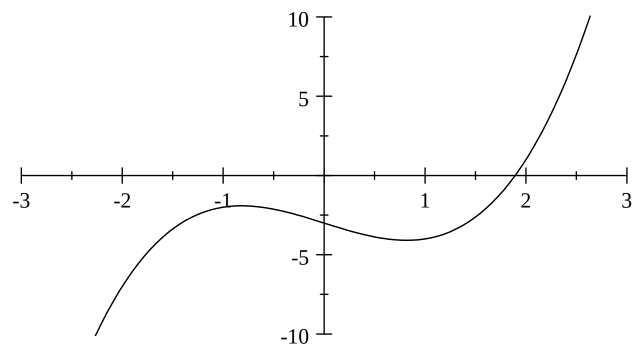
$f'(x) = 6x - 3x^2 = 3x(2-x)$. At a critical point, $f'(x) = 0$. Solving $3x(2-x) = 0$ gives $x = 0$ or $x = 2$. Therefore the critical points are $x = 0$ and $x = 2$.

The second derivative of f is

$$f''(x) = 6 - 6x.$$

Since $f''(0) = 6$ which is positive, $x = 0$ gives a point of relative minimum. On the other hand, $f''(2) = 6 - 12 = -6$ which is negative. Therefore $x = 2$ gives a point of relative maximum.

9. The graph below of the function $f(x) = x^3 - 2x - 3$ suggests that $x_0 = 2$ is an approximate solution of the equation $x^3 - 2x - 3 = 0$



Use Newton's method to obtain better approximate solution of the equation which is accurate to two decimal places.

A better approximate solution is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{8-4-3}{12-4} = 1.875$$

We do not know the level of accuracy of this approximation therefore we compute another approximation which is

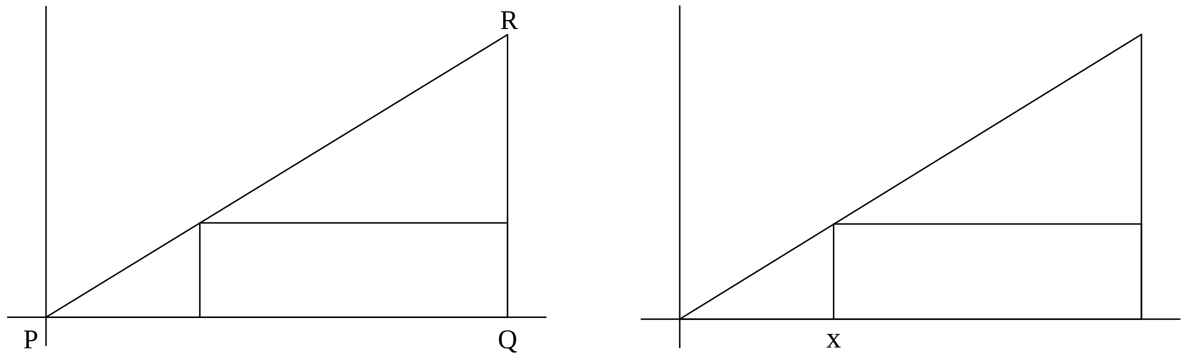
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.875 - \frac{f(1.875)}{f'(1.875)} = 1.875 - \frac{1.875^3 - 2(1.875) - 3}{3(1.875)^2 - 2} = 1.893510055$$

It is not obvious that this is accurate to 2 decimal places because x_1 and x_2 differ in the second decimal place, therefore we compute another approximation which is

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.893510055 - \frac{f(1.893510055)}{f'(1.893510055)} \\ &= 1.893510055 - \frac{1.893510055^3 - 2(1.893510055) - 3}{3(1.893510055)^2 - 2} = 1.893289228 \end{aligned}$$

Since there is no difference between the first 3 decimal places of x_2 and x_3 we are confident that x_3 is accurate to at least 3 decimal places. Therefore , to two decimal places, the solution is $x = 1.89$

10. A triangle has vertices at $P\,(0,0)$, $Q\,(6,0)$ and $R\,(6,5)$ and a rectangle is drawn inside the triangle as shown in the diagram.



(a) (b)

(a) The line segment PR passes through $(0,0)$ and $(6,5)$ therefore its equation i given by

$$\frac{y-0}{x-0} = \frac{5-0}{6-0} = \frac{5}{6}. \text{ This may be rearranged as } y = \tfrac{5}{6}x.$$

(b) The width of the rectangle in figure (b) is $\frac{5}{6}x$ and its length is $(6-x)$, (because Q has coordinates $(6,0)$. Therefore the area of the rectangle is

$$A = \text{ Length } \times \text{ Width } = \tfrac{5}{6}x\,(6-x)$$

(c) To determine the value of x that gives the area of the rectangle, we look for the critical point(s) of the function

$$A(x) = \tfrac{5}{6}x\,(6-x) = 5x - \tfrac{5}{6}x^2$$

Its derivative is $A'(x) = 5 - \frac{5}{3}x$, therefore $A'(x) = 0$ when $5 - \frac{5}{3}x$. Solving gives $x = 3$. Since $A''(x) = -\frac{5}{3}$ it follows that $A''(0) = -\frac{5}{3}$ which is negative. Therefore $x = 3$ corresponds to a maximum area of the rectangle.

11. Given the function $f(x) = \sqrt{5+4x}$ and the point $c = 1$, we have to determine the 3rd degree Taylor polynomial for f at the point $c = 1$. We have to evaluate $f(1)$, $f'(1)$, $f''(1)$ and $f'''(1)$.

$$f(1) = \sqrt{5+4} = \sqrt{9} = 3$$

$$f'(x) = \tfrac{1}{2}\,(5+4x)^{-\frac{1}{2}}\,(4) = 2\,(5+4x)^{-\frac{1}{2}}, \quad \text{therefore} \quad f'(1) = \frac{2}{3}$$

$$f''(x) = 2\left(-\tfrac{1}{2}\right)(5+4x)^{-\frac{3}{3}}\,(4) = -4\,(5+4x)^{-\frac{3}{3}}, \quad \text{therefore} \quad f''(1) = -\frac{4}{9^{\frac{3}{2}}} = -\frac{4}{27}$$

$$f'''(x) = -4\left(-\tfrac{3}{2}\right)(5+4x)^{-\frac{5}{2}}\,(4) = 12\,(5+4x)^{-\frac{5}{2}}, \quad \text{therefore} \quad f'''(1) = \frac{12}{9^{\frac{5}{2}}} = \frac{12}{243} = \frac{4}{81}$$

It follows that the 3rd degree Taylor polynomial for f at the point $c = 1$ is

$$\begin{aligned} p_3(x) &= f(1) + f'(1)\,(x-1) + \tfrac{1}{2}f''(1)(x-1)^2 + \tfrac{1}{6}f'''(1)\,(x-1)^3 \\ &= 3 + \tfrac{2}{3}\,(x-1) + \tfrac{1}{2}\left(-\tfrac{4}{27}\right)(x-1)^2 + \tfrac{1}{6}\left(\tfrac{4}{81}\right)(x-1)^3 \\ &= 3 + \tfrac{2}{3}\,(x-1) - \tfrac{2}{27}(x-1)^2 + \tfrac{2}{243}\,(x-1)^3 \end{aligned}$$