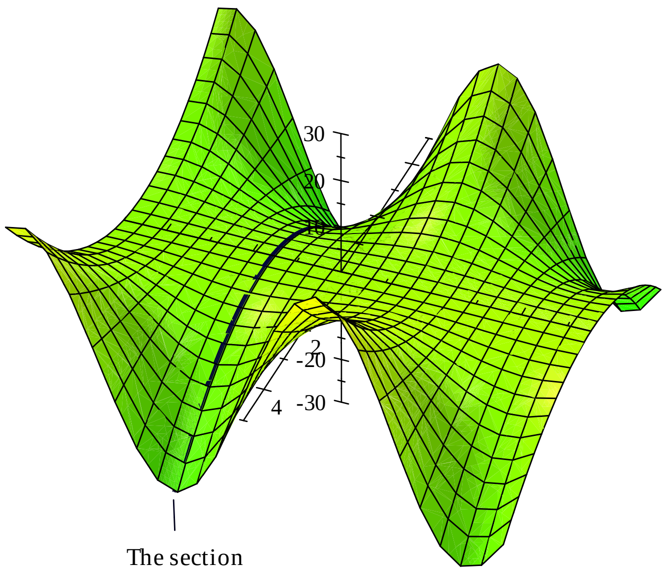
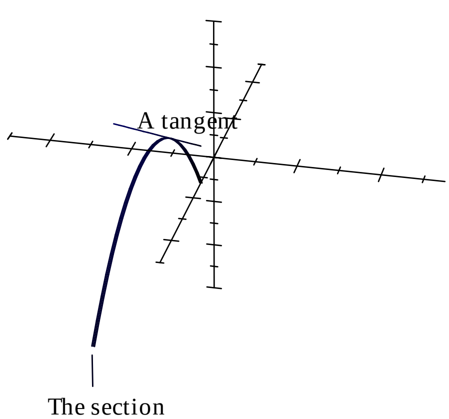
Partial Derivatives

Before defining derivatives of functions of several variables, it is necessary to introduce their partial derivatives. This is because the formulas for their derivatives involve formulas of their partial derivatives We introduce partial derivatives through an example. To this end, consider the function $f(x,y) = x + x^2 \sin y$. You should have encountered it in the exercises. Its graph is given below and we have also highlighted its "$y = -\frac{\pi}{2}$ section", (obtained by fixing y at $-\frac{\pi}{2}$ then vary x). The section is also drawn separately in the next figure.



Graphs of f and the "$y = -\frac{\pi}{2}$ section"

By definition, the $y = -\frac{\pi}{2}$ section is the curve $\mathbf{r}(x) =< x, -\frac{\pi}{2}, x + x^2 \sin\left(-\frac{\pi}{2}\right) > \quad = \quad < x, -\frac{\pi}{2}, x - x^2 >$ It is in the $y = -\frac{\pi}{2}$ plane that is parallel to the xz plane. For all practical purposes, we may take it to be the graph of $z = x - x^2$. It is shown below.



A tangent to the section at some point $\left(x, -\frac{\pi}{2}, x - x^2\right)$ on the curve is also included. The slope of such a tangent is called the partial derivative of f with respect to x at $(x, -\frac{\pi}{2})$ and it is denoted by $f_x(x, -\frac{\pi}{2})$ or $\dfrac{\partial f}{\partial x}(x, -\frac{\pi}{2})$. We pointed out that the section is practically the graph of the function $z = x - x^2$, therefore the slope of the tangent must be $1 - 2x$. Using the notation we introduced, $f_x(x, -\frac{\pi}{2}) = 1 - 2x$. We chose to fix y at $-\frac{\pi}{2}$ purely arbitrarily. If we fix it at an arbitrary number d, we get the $y = d$ section with formula

$$f(x, d) = x + x^2 \sin d$$

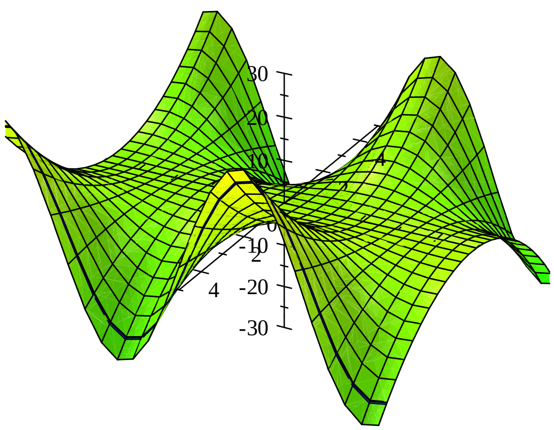
The slope of the tangent to its graph at a point $(x, d, f(x, d))$ is called the partial derivative of f with respect to x at (x, d) and it is denoted by $\dfrac{\partial f}{\partial x}(x, d)$ or $f_x(x, d)$. Its formula is

$$f_x(x, d) = 1 + 2x \sin d.$$

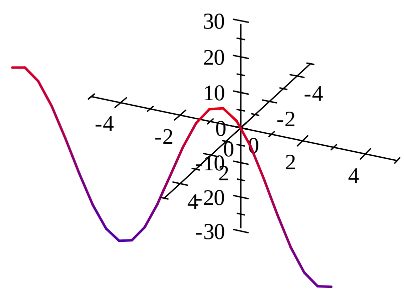
Partial derivatives with respect to y are defined in the same way. We get an x-section by fixing x and vary y. Say we fix x at 4.5 and vary y. We get points that lie on the curve

$$\mathbf{r}(y) = < 4.5, y, 4.5 + 4.5^2 \sin y >$$

It is highlighted in the graph to the left and drawn separately in the graph to the right.



Graphs of f and the "$x = 4.5$ section"



Graph of the "$x = 4.5$ section of f"

The slope of the tangent to the curve at $(4.5, y, f(4.5, y))$ is called the partial derivative of f with respect to y at $(4.5, y)$. It is denoted by $\dfrac{\partial f}{\partial y}(4.5, y)$ or $f_y(4.5, y)$. Since the curve is in a plane parallel to the yz plane, it is practically the graph of $z = 4.5 + 4.5^2 \sin y$. Therefore, the slope of the tangent at a point $(4.5, y, f(4.5, y))$ is $4.5^2 \cos y$, thus

$$f_y(4.5, y) = 4.5^2 \cos y$$

If we fix x at an arbitrary number c, we get the $x = c$ section with formula

$$f(c, y) = c + c^2 \sin y$$

The slope of the tangent to its graph at a point $(c, y, f(y, c))$ is called the partial derivative of f with respect to y at (c, y), denoted by $f_y(c, y)$ or $\dfrac{\partial f}{\partial y}(c, y)$. It has formula $f_y(x, d) = c^2 \cos y$.

With the above examples in the background, we turn to the partial derivatives of a general function $f(x, y)$ of two variables. Let (c, d) be a point in its domain. Consider the y-section obtained by fixing y at d and varying x. It is a function of one variable x and its graph is a curve

$$\mathbf{r}(x) = < x, d, f(x, d) >$$

The slope of the tangent to the curve at a point $(x, d, f(x, d))$ on the curve is called the partial derivative of f with respect to x at (x, d). It is denoted by $f_x(x, d)$ or $\frac{\partial f}{\partial x}(x, d)$ and it is obtained by simply computing, in the usual way, the derivative of the function $f(x, d)$ of one variable x.

Example 1 *Let $f(x, y) = x^2 + 3y^4 - 4e^{x^2 y}$. If we fix y at d we get the function $f(x, d) = x^2 + 3d^4 - 4e^{dx^2}$. Its derivative as a function of one variable x is the partial derivative of f with respect to x at (x, d) and it is given by $f_x(x, d) = 2x - 8dxe^{dx^2}$.*

In general, we will ask for the partial derivative of f with respect to x at a point (x, y) in the domain of f. It is then understood that you must regard y as a constant and determine the derivative of the function $f(x, y)$ of one variable x. In particular, if $f(x, y) = x^2 + 3y^4 - 4e^{x^2 y}$ is the function in the Example 1 then $f_x(x, y) = 2x - 8yxe^{x^2 y}$.

Example 2 *Let $f(x, y) = 7 + \frac{4x}{y^2} - 3\cos \pi xy^2 + x - y$. Then $f_x(x, y) = \frac{4}{y^2} + 3\pi y^2 \sin \pi xy^2 + 1$.*

The partial derivative with respect to y of an arbitrary function $f(x, y)$ is defined in a similar way. Thus we consider an x-section of f obtained by fixing x at c and varying y. It is a function of one variable y and its graph is a curve

$$\mathbf{r}(y) = < c, y, f(c, y) >$$

The slope of the tangent to this curve at a point $(c, y, f(c, y))$ is called the partial derivative of f with respect to y at (c, y). It is denoted by $f_y(c, y)$ or $\frac{\partial f}{\partial y}(c, y)$. It is obtained by simply computing, in the usual way, the derivative of the function $f(c, y)$ of one variable y.

Example 3 *Let $f(x, y) = x^2 + 3y^4 - 4e^{x^2 y}$. If we fix x at c, we get the function $f(c, y) = c^2 + 3y^4 - 4e^{c^2 y}$. Its derivative as a function of one variable y is the partial derivative of f with respect to y at (c, y) and it is given by $f_y(c, y) = 12y^3 - 4c^2 e^{c^2 y}$.*

In general, we will ask for the partial derivative of f with respect to y at point (x, y) in the domain of f. It is then understood that you must regard x as a constant and determine the derivative of the function $f(x, y)$ of one variable y. For example, if $f(x, y) = x^2 + 3y^4 - 4e^{x^2 y}$ is the function in example 1 then $f_y(x, y) = 12y^3 - 4x^2 e^{x^2 y}$.; and if $f(x, y) = 7 + \frac{4x}{y^2} - 3\cos \pi xy^2 + x - y$ is the function in Example 2 then $f_y(x, y) = -\frac{8x}{y^3} + 6\pi xy \sin \pi xy^2 - 1$.

Exercise 4

1. *Let $f(x, y) = xy + x^2y^3 - x^3 + 3y$. Verify that $f_x(x, y) = y + 2xy^3 - 3x^2$ and $f_y(x, y) = x - 3x^2y^2 + 3$*
2. *Complete the table*

$f(x, y)$	$f_x(x, y)$	$f_y(x, y)$
$x^2 - y^3 + xy - x - y$		
$x^4y^3 + 5xy - x + 2y$		
$\frac{4x}{3y} - \frac{5y}{7x}$		
$4xe^y - 7\tan 2y$		
$x \sin y - y^2 \sin 4x$		
$\sin xy + \cos x - 3 \sin y$		

$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$
$\sin x \cos y - \sin xy$		
$4e^x \sin 2y - 5\tan xy$		
$xe^{3xy} + y^3$		
$e^x \cos y + 15x$		
$4x \arcsin 2y$		
$5 - 7x \arctan y$		
$\dfrac{x+2y}{3x-y}$		
$6e^{\frac{4x}{y}} - \dfrac{y}{4x}$		

Partial Derivatives for Functions of Three or More Variables

Since we cannot draw graphs of function of three or more variables in 3-dimensional space, their partial derivatives are defined without reference to graphs. Given a function $f(x,y,z)$ of 3 variables x, y and z. its partial derivative with respect to any one of them is obtained by fixing the other two variables then take the derivative of the resulting function of one variable x. For example, given $f(x,y,z) = z^3 - \dfrac{4y}{3x} + xyz$, its partial derivative with respect to x, denoted by f_x or $\dfrac{\partial f}{\partial x}$, is obtained by fixing y and z then take the derivative of the function $x \to z^3 - \dfrac{4y}{3x} + xyz$ of one variable x. Thus

$$f_x(x,y,z) = \frac{4y}{3x^2} + yz.$$

Its partial derivatives with respect to y and z are denoted by f_y or $\dfrac{\partial f}{\partial y}$ and f_z or $\dfrac{\partial f}{\partial z}$ respectively and they are

$$f_y(x,y,z) = -\frac{4}{3x} + xz, \qquad f_z(x,y,z) = 3z^2 + xy.$$

Exercise 5 *Complete the following table:*

$f(x,y,z)$	$f_x(x,y,z)$	$f_y(x,y,z)$	$f_z(x,y,z)$
$4xyz - 3x^4y^2z^3$			
$z^2 \sin xy$			
$e^{xy} \tan \pi yz$			
$y \ln\left(1 + x^2z^2y\right)$			

Formal Definitions of Partial Derivatives

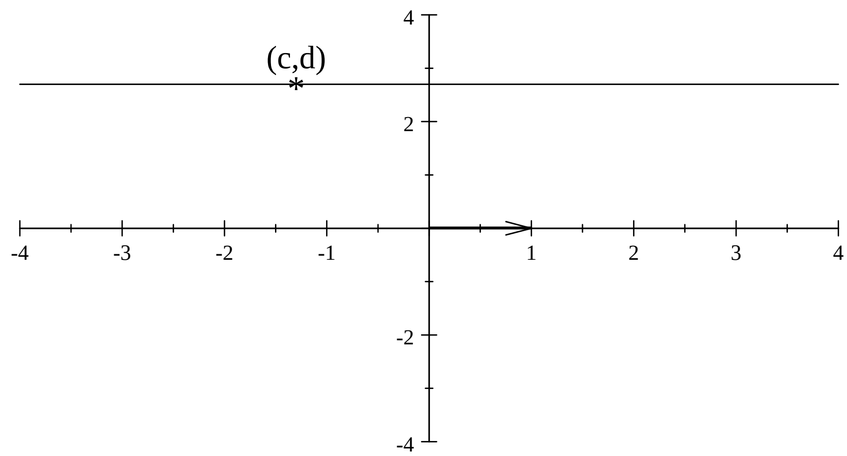
Let $f(x,y)$ be a function of two variables and (c,d) be a point in its domain. To determine $f_x(c,d)$, we keep y fixed at d and vary x. The result is a function of one variable x which we may denote by $u(x)$. Its derivative at c is the number

$$u'(c) = \lim_{h \to 0} \frac{u(c+h) - u(c)}{h} = \lim_{h \to 0} \frac{f(c+h,d) - f(c,d)}{h}$$

Therefore

$$f_x(c,d) = \lim_{h\to 0} \frac{f(c+h,d) - f(c,d)}{h} \tag{1}$$

Note that when we keep y fixed at d and vary x, we get the set of points $\{(x,d) : x \text{ is a real number}\}$ which lie on the straight line through (c,d), parallel to the vector $\mathbf{i}$.



A line segment through (c,d) parallel to the vector $\mathbf{i}$.

This implies that $f_x(c,d) = \lim_{h\to 0} \frac{f(c+h,d) - f(c,d)}{h}$ is the rate of change of f along the line segment through (c,d) parallel to $\mathbf{i}$. For this reason, $f_x(c,d)$ is also called the directional derivative of f, (or simply the rate of change of f), at (c,d) in the direction of the vector $\mathbf{i}$.

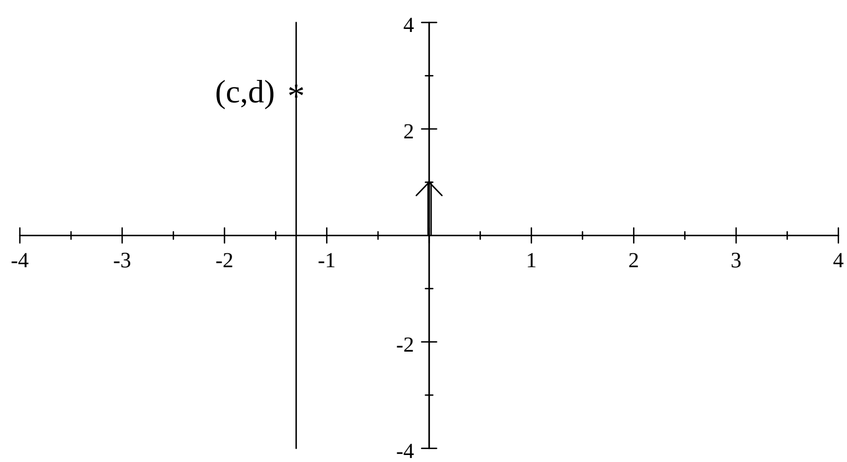
To determine $f_y(c,d)$, we fix x at c and vary y, to get a function $v(y) = f(c,y)$ of one variable y. Its derivative at d is the number

$$v'(d) = \lim_{k\to 0} \frac{v(d+k) - u(d)}{k} = \lim_{k\to 0} \frac{f(c,d+k) - f(c,d)}{k}$$

Therefore

$$f_y(c,d) = \lim_{k\to 0} \frac{f(c,d+k) - f(c,d)}{k} \tag{2}$$

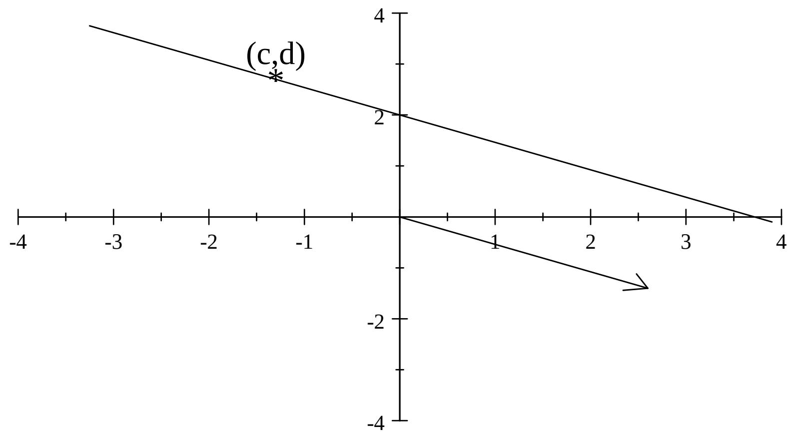
This time, when we keep x fixed at c and vary y, we get the set of points $\{(c,y) : y \text{ is a real number}\}$ which lie on the straight line that passes through (c,d) and is parallel to the vector $\mathbf{j}$.



A line segment through (c,d) parallel to the vector $\mathbf{j}$.

Therefore $f_y(c,d) = \lim_{k\to 0} \frac{f(c,d+k)-f(c,d)}{k}$ is the rate of change of f along the line segment through (c,d) parallel to $\mathbf{j}$. Consequently, $f_y(c,d)$ is also called the directional derivative of f, (or simply the rate of change of f), at (c,d) in the direction of the vector $\mathbf{j}$.

We may generalize the above directional derivatives by considering the directional derivative of f at (c,d) in the direction of an arbitrary nonzero vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$.



This requires us to evaluate f on a line segment through (c,d), parallel to the vector $\mathbf{u}$. The points on the line segment have the form $(c+u_1t, d+u_2t)$ where t is any real number. Therefore we must compute the limit of $\frac{f(c+u_1t,d+u_2t)-f(c,d)}{t}$ as t tends to 0, denoted by $D_{\mathbf{u}}f(c,d)$.

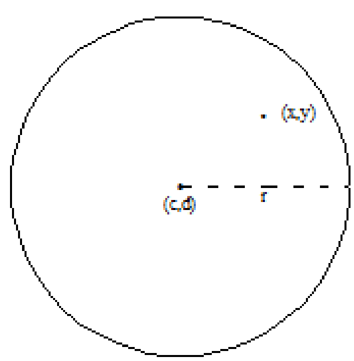
Example 6 *Let $f(x,y) = 3x^2y$, (c,d) be a point in the plane and $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ be a given vector. To calculate the rate of change of f at (c,d) in the direction of $\mathbf{u}$, we evaluate f at points $(c+u_1t, d+u_2t)$ on a line segment through (c,d) parallel to $\mathbf{u}$ then compute the following limit:*

$$\lim_{t\to 0} \frac{f(c+u_1t,d+u_2t)-f(c,d)}{t}$$

Direct computations reveal that $f(c+u_1t,d+u_2t)-f(c,d) = t\left(6u_1cd + 3c^2u_2 + u_1^2td + 2u_1u_2ct + u_1u_2t^2\right)$. Therefore

$$D_{\mathbf{u}}f(c,d) = \lim_{t\to 0} \frac{f(c+u_1t,d+u_2t)-f(c,d)}{t} = 6u_1cd + 3c^2u_2$$

If the formula for f is not specified, we have to appeal to the Mean Value Theorem to determine an expression for its directional derivatives. To this end, let f be an arbitrary function of two variables and (c,d) be a point in its domain. We have to insist that its partial derivatives f_x and f_y are continuous on some disc centred at (c,d). Intuitively, this means that if (x,y) is in some sufficiently small disc, of radius r, centred at (c,d) then $f_x(x,y)$ is approximately equal to $f_x(c,d)$ and $f_y(x,y)$ is approximately equal to $f_y(c,d)$.



More precisely, given any $\varepsilon > 0$ we can find a disc of radius r with the property if (x, y) is any point in the disc then

$$f_x(c,d) - \varepsilon < f_x(x,y) < f_x(c,d) + \varepsilon \qquad \text{and} \qquad f_y(c,d) - \varepsilon < f_y(x,y) < f_y(c,d) + \varepsilon \tag{3}$$

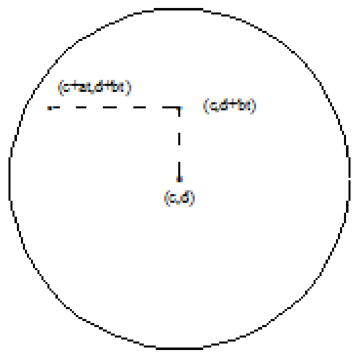
For simplicity of notation, denote the given vector by $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ (instead of $u_1\mathbf{i} + u_2\mathbf{j}$). Then the derivative of f at (c, d) in the direction of $\mathbf{u}$ is

$$\lim_{t \to 0} \frac{f(c + at, d + bt) - f(c,d)}{t}.$$

We may consider only those numbers t such that $(c + at, d + bt)$ is in the above disc of radius r centred at (c, d). To use the Mean Value Theorem, we write $f(c + at, d + bt) - f(c, d)$ as

$$f(c + at, d + bt) - f(c, d + bt) + f(c, d + bt) - f(c, d)$$

The three points $(c + at, d + bt)$, $(c, d + bt)$ and (c, d) are shown in the figure below.



Consider the term $f(c + at, d + bt) - f(c, d + bt)$. If we introduce the function $w(x) = f(c + x, d + bt)$ then

$$f(c + at, d + bt) - f(c, d + bt) = w(at) - w(0)$$

Note that $w(x)$ is a function of one variable x. By the Mean Value Theorem, there is a number θ between 0 and at such that

$$w(at) - w(0) = (at - 0)\, w'(\theta) = atw'(\theta)$$

It turns out that $w'(\theta) = f_x(\theta, d+bt)$ and, because $(\theta, d+bt)$ is in the above disc, $f_x(\theta, d+bt)$ is approximately equal to $f_x(c, d)$. Therefore

$$f(c + at, d + bt) - f(c, d + bt) \simeq atf_x(c, d) \tag{4}$$

If you want to be more rigorous than this, use (3) to deduce that $atw'(\theta)$ is between $atf_x(c, d) - |at|\, \varepsilon$ and $atf_x(c, d) + |at|\, \varepsilon$.

Handle $f(c, d + bt) - f(c, d)$ in a similar way to deduce that

$$f(c, d + bt) - f(c, d) \simeq btf_y(c, d) \tag{5}$$

It follows from (4) and (5) that

$$\frac{f(c + at, d + bt) - f(c, d)}{t} \simeq \frac{atf_x(c, d) + btf_y(c, d)}{t} = af_x(c, d) + bf_y(c, d)$$

This implies that

$$\lim_{t \to 0} \frac{f(c + at, d + bt) - f(c, d)}{t} = af_x(c, d) + bf_y(c, d)$$

In other words, if f has continuous partial derivatives then its directional derivative at (c, d) in the direction of a vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is

$$D_{\mathbf{u}} f(c, d) = af_x(c, d) + bf_y(c, d) = a\frac{\partial f}{\partial x}(c, d) + b\frac{\partial f}{\partial y}(c, d)$$

When we apply this result to the function $f(x,y) = 3x^2y$ of Example 6 above we conclude that its directional derivative at a point (c,d) in the direction of a vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$u_1 f_x(c,d) + u_2 f_y(c,d) = u_1(6cd) + u_2\left(3c^2\right) = 6u_1cd + 3u_2c^2$$

as we obtained directly.

For convenience, introduce the vector $f_x\mathbf{i} + f_y\mathbf{j}$. Then the directional derivative of f at (c,d) in the direction of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ may be written as the scalar product

$$D_{\mathbf{u}}f(c,d) = (f_x\mathbf{i} + f_y\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$$

You are going to run into $f_x\mathbf{i} + f_y\mathbf{j}$ ahead. It is given a special symbol which is $\boldsymbol{\nabla} f$, pronounced "Grad f", (short for the gradient of f). Thus, by definition

$$\boldsymbol{\nabla} f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$$

Exercise 7

1. *Let* $f(x,y) = x^3y + y^2x - 3$. *Use the definition:*

   (a) $f_x(x,y) = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$ *to calculate* $f_x(x,y)$.

   (b) $f_y(x,y) = \lim\limits_{k\to 0} \dfrac{f(x,y+k) - f(x,y)}{k}$ *to calculate* $f_y(x,y)$.

   (c) $D_{\mathbf{u}}f(x,y) = \lim\limits_{t\to 0} \dfrac{f(x+u_1t, y+u_2t) - f(x,y)}{t}$, *the directional derivative of* f *at* (x,y) *in the direction of* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$.

2. *Find* f_x *and* f_y *given that* $f(x,y) =$

a) $5 - xy + 2x^2y^2 - 3x^3y^4$	*b)* $x\sin y - y\cos x + 5xy$	*c)* $2ye^{xy} - 3x\sin xy - 3x + 2y$
d) $4e^x - x^2\ln(2xy+1)$	*e)* $4x - xy\cos 3xy$	*f)* $e^x\sin xy - y^2\ln\left(2x^2y+1\right) - \dfrac{y^2}{4-x}$

3. *Let* $f(x,y)$ *be a given function,* $\left\{(x,y,f(x,y)) : (x,y) \in \mathbb{R}^2\right\}$ *be its graph and* $(c,d,f(c,d))$ *be a point on the graph. Then* $\{(x,d,f(x,d)) : x \in \mathbb{R}\}$ *and* $\{(c,y,f(c,y)) : y \in \mathbb{R}\}$ *are curves in the graph of* f. *Determine tangents to these curves at* $(c,d,f(c,d))$ *and use them to show that* $-f_x(c,d)\mathbf{i} - f_y(c,d)\mathbf{j} + \mathbf{k}$ *is a normal to the tangent plane at* $(c,d,f(c,d))$. *(This result is used in several parts ahead.) Now show that the equation of the tangent plane to the graph of* $f(x,y)$ *at* $(c,d,f(c,d))$ *is*

$$z = (x-c)\,f_x(c,d) + (y-d)f_y(c,d) + f(c,d)$$

4. *Calculate* $\boldsymbol{\nabla} f$ *given that* $f(x,y) = \sin 2x + \cos 3y - 4x^2y^3$.

5. *The gradient of a function* $f(x,y,z)$ *of three variables is also denoted by* $\boldsymbol{\nabla} f$ *and it is defined, as you would expect, by*

$$\boldsymbol{\nabla} f = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

   Calculate $\boldsymbol{\nabla} f$ *given that* $f(x,y,z) =$

   a) $x^2yz + y^2xz - z^2xy$ *b)* $x\ln\left(x + y^2z^3\right)$ *c)* ze^{xy}

Remark 8 *In a number of applications,* $\boldsymbol{\nabla}$ *is regarded as a vector "operator" with* $\mathbf{i}$ *component* $\dfrac{\partial}{\partial x}$, $\mathbf{j}$ *component* $\dfrac{\partial}{\partial y}$ *and* $\mathbf{k}$ *component* $\dfrac{\partial}{\partial z}$. *It "operates" on a given function* f *to give a vector*

$$\boldsymbol{\nabla} f = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} > = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}.$$

Higher Order Partial Derivatives

As an example, consider the function $f(x, y) = x\cos y - 4x^3y^2 + 7x - 2$. Its partial derivative with respect to x is $f_x(x, y) = \cos y - 12x^2y^2 + 7$ and its partial derivative with respect to y is $f_y(x, y) = -x\sin y - 8x^3y$. These two are called the first order partial derivatives of f.

Since f_x and f_y are functions of x and y, we may consider calculating their partial derivatives. Take $f_x(x, y) = \cos y - 12x^2y^2 + 7$. Its partial derivative with respect to x is denoted by $(f_x)_x(x, y)$, which is shortened to $f_{xx}(x, y)$, or $\dfrac{\partial^2 f}{\partial x^2}(x, y)$. Therefore

$$f_{xx}(x, y) = 24xy^2$$

Its partial derivative of $f_x(x, y)$ with respect to y is denoted by $(f_x)_y(x, y)$, which is shortened to $f_{xy}(x, y)$. Another notation is $\dfrac{\partial^2 f}{\partial x \partial y}(x, y)$. Therefore

$$f_{xy}(x, y) = -\sin y - 24x^2y$$

Likewise, the partial derivative of $f_y(x, y) = -x\sin y - 8x^3y$ with respect to x is denoted by $f_{yx}(x, y)$ or $\dfrac{\partial^2 f}{\partial y \partial x}(x, y)$. It is

$$f_{yx}(x, y) = -\sin y - 24x^2y$$

The partial derivative of f_y with respect to y is denoted by $f_{yy}(x, y)$ or $\dfrac{\partial^2 f}{\partial y^2}(x, y)$. Thus

$$f_{yy}(x, y) = -x\cos y - 8x^3$$

The four functions $f_{xx}(x, y)$, $f_{xy}(x, y)$, $f_{yx}(x, y)$ and $f_{yy}(x, y)$ are called the second order partial derivatives of f. The middle two, i.e. $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are called its mixed order partial derivatives. Note that, in this particular case, they are equal. For functions like this one, which have continuous partial derivatives, that will always be the case.

In general, given a function $f(x, y)$ of two variables, we may determine the partial derivatives of f_x with respect to x or y and the partial derivatives of f_y with respect to x or y.

The partial derivative of f_x with respect to x is denoted by f_{xx} or $\dfrac{\partial^2 f}{\partial x^2}(x, y)$ and that with respect to y is denoted by f_{xy} or $\dfrac{\partial^2 f}{\partial x \partial y}(x, y)$.

The partial derivative of f_y with respect to x is denoted by f_{yx} or $\dfrac{\partial^2 f}{\partial y \partial x}(x, y)$ and that with respect to y is denoted by f_{yy} or $\dfrac{\partial^2 f}{\partial y^2}(x, y)$.

The four functions f_{xx}, f_{xy}, f_{yx} and f_{yy} are called the second order partial derivatives of f. If f_x and f_y are continuous then $f_{xy} = f_{yx}$.

Exercise 9 *Determine all the second order partial derivatives of f given that $f(x, y) =$*

a) $x^2 - 3xy + 2y^2 + 5x$ *b)* $5\cos x \sin y$ *c)* $x\cos y - y\cos x$

d) $x^3 + x^2y - 3y^2 + 2y^3$ *e)* $4e^x \cos y$ *f)* $2e^{xy}(2\sin x - 3\cos y)$