

Review of Exponents

There are instances when we have to multiply a given number by itself several times. We use exponents to write such products more briefly. Examples:

$2 \times 2 \times 2 \times 2 \times 2$ is written briefly as 2^5 . By agreement, 2 is called the base and 5 is called the exponent.

$(-6)(-6)(-6)$ is written briefly as $(-6)^3$. The base is -6 and the exponent is 3.

$y \times y \times y \times y \times y \times y$ is written briefly as y^6 . The base is y and the exponent is 6.

$(a-b)^4$ means $(a-b)(a-b)(a-b)(a-b)$. The base is $(a-b)$ and the exponent is 4.

Exponents can be negative. By agreement, a negative exponent means the reciprocal of the corresponding positive exponent. For example, 8^{-4} means

$$\frac{1}{8^4} \quad \text{which is a brief way of writing} \quad \frac{1}{8 \times 8 \times 8 \times 8}.$$

In general, if n is a given positive integer and b is a non-zero number then b^{-n} means $\frac{1}{b^n} = \frac{1}{b \times b \times b \times \dots \times b}$ where the number of b 's in the denominator is n .

When exponents appear with one or more of the multiplication, division, addition and subtraction operations, *and if there are no parentheses to dictate the order of operations, then exponents are performed first, followed by multiplications or divisions from left to right and then addition or subtractions.* For example, to evaluate $13 - 6^2 \div 4 \times 7 + 24 \div 2^3 + 9 - 5$, we first do the exponents to get

$$13 - 36 \div 4 \times 7 + 24 \div 8 + 9 - 5.$$

We then do the multiplications and divisions from left to right and the result is

$$13 - 63 + 3 + 9 - 5$$

Finally, we do the additions and subtractions and the final result is -43 .

Calculators that record the operations to be performed are programmed to implement these procedures. For example, if you input

$$17 \div 15 \times 75 + 12$$

in such a calculator then press the = button, it will display the answer 97 which is the result dividing 17 by 15 then multiply the result by 75 and then add 12.

A common mistake among calculator users

This is generally made when evaluating a quotient like $\frac{56}{14+7}$ in which the denominator contains more than one term. Some individuals input this in a calculator as $56 \div 14 + 7$ which comes to $4 + 7 = 11$. **But by agreement,** $\frac{56}{14+7}$ means the result of dividing 56 by the sum of 14 and 7 which is $\frac{56}{21} = \frac{8}{3}$. To avoid such errors, regard a quotient $\frac{p}{q}$ as $p \div (q)$. Thus

$$\frac{56}{14+7} = 56 \div (14+7) = 56 \div 21 = \frac{8}{3}$$

$$\frac{90}{3 \times 4 + 6} = 90 \div (3 \times 4 + 6) = 90 \div 18 = 5, \text{ NOT } 90 \div 3 \times 4 + 6, \text{ (which simplifies to 126).}$$

$$\frac{96}{8 \times 4} = 96 \div (8 \times 4) = 96 \div 32 = 3, \text{ NOT } 96 \div 8 \times 4, \text{ (which simplifies to 48).}$$

Rules of Exponents

Most probably you are familiar with the following properties of exponents:

1. For any nonzero base a and nonzero exponent n ,

$$\frac{1}{a^{-n}} = a^n$$

2. For any nonzero base a ,

$$a^0 = 1$$

3. When two or more expressions with same base are multiplied together, then the result is the same base with sum of the exponents. Thus

$$a^m a^n a^k = a^{m+n+k}$$

4. When an exponent is raised to a power, the result is the base raised to the product of the exponents. In other words

$$(a^m)^n = a^{mn}.$$

5. When an *exponential expression is divided by another exponential expression* with the same base, then the result is an exponential expression with the same base raised to the difference between the exponent of the numerator and the exponent of the denominator. More precisely,

$$\frac{a^m}{a^n} = a^{m-n}$$

6. If there are no parentheses to dictate the order of operation, an exponent of an exponent is, by agreement, evaluated from "below". For example, to evaluate 2^{3^4} , you start by evaluating 2^3 to get 8. You then raise 8 to power 4 to get 4096. You do not start by evaluating 3^4 to get 81 then raise 2 to power 81. The instructions to do that would be written as $2^{(3^4)}$.

Examples

1. $3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$

2. $\left(\frac{2}{3}\right)^4 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{16}{81}$ NOTE: Since we do exponentiations before multiplying, we cannot write $\left(\frac{2}{3}\right)^4$ as $\frac{2^4}{3}$ because the latter means $\frac{16}{3}$, (exponentiate to get $2^4 = 16$ then divide by 3).

3. $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$ NOTE: Since we do exponentiations before multiplying, $(-2)^4$ is not the same as -2^4 . In fact -2^4 means the negative of 2^4 which is -16 .

4. $\left(-\frac{1}{5}\right)^3 = \left(-\frac{1}{5}\right) \times \left(-\frac{1}{5}\right) \times \left(-\frac{1}{5}\right) = -\frac{1}{125}$

5. $3 \times (4)^2 = 3 \times 4 \times 4 = 3 \times 16 = 48$ NOTE: Since we do exponentiations before multiplying, $3 \times (4)^2$ may also be written more simply as 3×4^2 . It is understood that you first square 4 then multiply the result by 3. It does not mean the square of 3×4 .

6. $-2(3)^4 = -2(3 \times 3 \times 3 \times 3) = -2(81) = -162$

7. $(2)^{-3} = \frac{1}{2^3} = \frac{1}{8}$ NOTE: $(2)^{-3}$ may be written more simply as 2^{-3} .

8. $\left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^2} = \frac{1}{\left(\frac{4}{9}\right)} = \frac{9}{4}$

$$9. \left(\frac{3}{2}\right)^2 - 4^{-2} = \left(\frac{3}{2}\right) \times \left(\frac{3}{2}\right) - \frac{1}{4^2} = \frac{9}{4} - \frac{1}{16} = \frac{36}{16} - \frac{1}{16} = \frac{35}{16}$$

$$10. (-2)^0 = 1$$

$$11. 4 - 7^0 = 4 - 1 = 3$$

$$12. 1.25^2 = (1.25) \times (1.25) = 1.5625$$

$$13. 1.5^{-2} = \frac{1}{1.5^2} = \frac{1}{2.25}$$

14. To evaluate $(2x^3y^4)^5$, raise every term in parentheses to power 5 then simplify:

$$(2x^3y^4)^5 = (2)^5 (x^3)^5 (y^4)^5 = 32x^{15}y^{20}$$

$$15. (-4xy^3)(-3x^6y^5) = (-4)(-3)(x \times x^6)(y^3 \times y^5) = 12x^7y^8$$

$$16. \frac{65x^3y^8}{-13x^{-4}y^5} = \left(\frac{65}{-13}\right)(x^3x^4)\left(\frac{y^8}{y^5}\right) = -5x^7y^3$$

17. To evaluate $\left(\frac{3x^3}{y^5}\right)^{-4}$, raise every term in parentheses to power -4 then simplify:

$$\left(\frac{3x^3}{y^5}\right)^{-4} = \frac{3^{-4}(x^3)^{-4}}{(y^5)^{-4}} = \frac{3^{-4}x^{-12}}{y^{-20}} = \frac{y^{20}}{3^4x^{12}} = \frac{y^{20}}{81x^{12}}$$

18. To evaluate 3^{2^3} , we first determine 3^2 which is 9. We then evaluate 9^3 . Therefore $3^{2^3} = 9^3 = 729$.

19. To evaluate 2^{3^4} , we first determine 2^3 which is 8. We then determine 8^4 which turns out to be 4096. We finally determine 4096^2 which turns out to be 16777216. Therefore

$$2^{3^4} = 8^4 = 4096^2 = 16777216$$

Exercise 1

1. Simplify each expression. Your answers should NOT contain negative exponents.

$$a. 3^5$$

$$b. -2^6$$

$$c. 3^2 - (-2)^3$$

$$d. \left(-\frac{2}{3}\right)^{-4}$$

$$e. \left(-\frac{3}{4}\right)^3 - 2^{-3}$$

$$f. 4 + 7x^8 - 5x(2^3)$$

$$g. 3^{-2} \times 5$$

$$h. 2.25^3$$

$$i. \frac{(2x)^4}{3} - \left(\frac{2x}{3}\right)^4$$

$$j. 3x^{-2}y^3x^6z^0$$

$$k. \frac{5a^3b^{-4}c^{-8}}{25a^{-1}b^5c^0}$$

$$l. 7^{-1}p^{-6}q^8r^3p^3$$

$$m. \frac{36a^{15}b^7}{-6a^5b^3}$$

$$n. \left(\frac{x^3y^2z^8}{x^{-6}y^{-5}z^{-3}}\right)^{-3}$$

$$p. \left[\frac{4a^3b^{-2}}{c^2(2c^{-1}b^2)^3}\right]^{-4}$$

$$q. \frac{(2xy)^3z}{(-6x^2yz)^2}$$

$$r. \frac{27a^{-7}b^5c^0}{18a^{-5}b^3c}$$

$$s. x^2^3x^{-5}y^{-8}y^{3^4} + x - y^3$$

2. Point out the error(s) in the given computation then give the correct answer:

$$(a) 5 - 3x + 9x = 2x + 9x = 11x$$

$$(b) -3^2 + 6 - 0.8 = 9 + 6 - 0.8 = 14.2$$

(c) $5 \times 2^3 + 4 - 3y = 1000 + y$

3. Which of the two numbers 3^4 and 4^3 is bigger, and by how much?

4. Show that if x is any non-zero number then:

(a) $x^{2^{3^4}} = x^{2^4}$

(b) $x^{3^{2^3}} = x^{18}$

(c) $x^{m^{n^p}} = x^{(mnp)}$ for any integers m, n, p .

5. The number of bacteria in a culture doubles every 8 hours. Say you start with 100 bacteria. How many will there be: (a) 1 day later, (b) 2 days later, (c) 3 days later, (d) n days later?

Scientific Notation

Scientific notation is a way of writing numbers that is extremely convenient when handling very large or very small numbers. All numbers are written in the form

$$a \times 10^n$$

where a is a number between -10 and 10 and n is a whole number which may be positive or negative. Examples:

Ordinary decimal notation	Scientific notation	Some calculators display this as
401	4.01×10^2	4.01E2
5,998,000	5.998×10^6	5.998E6
0.0027	2.7×10^{-3}	2.7E - 3
0.0000000007554	7.554×10^{-10}	7.554E - 10
-781000	-7.81×10^5	-7.81E5

The E2 in 4.01E2 may be viewed as an abbreviation for **10 To Exponent 2**. Likewise the E - 3 in 2.7E - 3 may be viewed as an abbreviation for **10 To Exponent -3**.

Exercise 2

1. Convert into scientific notation:

(a) 4985 (b) -64800 (c) 25270000 (d) 0.0609 (e) -0.0000357 (f) -0.0002003

2. Convert into ordinary decimal notation:

(a) 3.55×10^4 (b) 5.102×10^8 (c) -7.311×10^{-3} (d) 1.08×10^{-7} (e) -4.4×10^3

3. Evaluate each expression and write your final answer in scientific notation. (There is no need for a calculator.)

(a) $(4.5 \times 10^4)(4 \times 10^3)$ (b) $\frac{-3.2 \times 10^8}{6.4 \times 10^5}$ (c) $\frac{8.4 \times 10^{-5}}{(5 \times 10^3)(2 \times 10^{-7})}$

4. The mass of one oxygen molecule is 5.3×10^{-23} gram. Find the mass of 35,000 molecules and write your answer in scientific notation.

5. The mass of a hydrogen atom is $1.67(10)^{-24}$ gram. Calculate the mass of 65,000 hydrogen atoms and write your answer in scientific notation.

6. The mass of the sun is given as 1.98892×10^{30} Kilograms. It is estimated that this is about 332,950 times the mass of the Earth and 1,048 times the mass of Jupiter. Use this information to estimate the masses of the Earth and Jupiter. Give your answers in scientific notation.

7. Assume the population of the United States is 3.12×10^8 , and the national debt is 1.6×10^{12} dollars. Find how much the share of national debt is for each person.