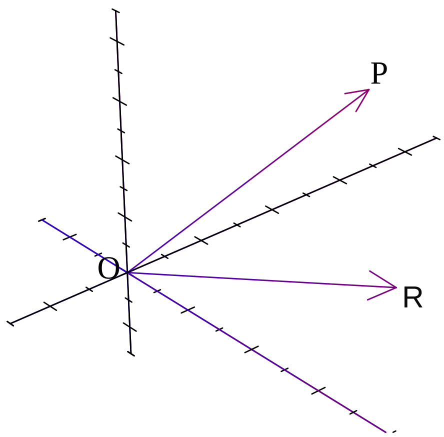
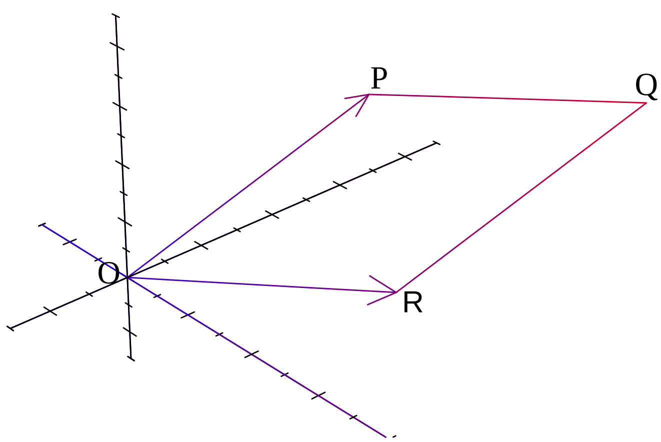
The Cross Product of Two Vectors

In proving some statements involving surface integrals, there will be a need to approximate areas of segments of the surface by areas of parallelograms. Therefore it is useful to introduce a method of computing areas of parallelograms in space. For the area of a parallelogram determined by two vectors, we use the "*cross product*" of the vectors. To define this, consider vectors $\overrightarrow{OP} = \mathbf{u} =< u_1, u_2, u_3 >$ and $\overrightarrow{OR} = \mathbf{v} =< v_1, v_2, v_3 >$ shown below.



Form the parallelogram $OPQR$ which has $\mathbf{u}$ and $\mathbf{v}$ as two of its sides and let A be its area.



Let the angle between the two given vectors be θ. We may assume that $0° < \theta < 180°$. From our knowledge of areas of parallelograms

$$A = ||\mathbf{u}||\ ||\mathbf{v}|| \sin\theta \tag{1}$$

The definition of the dot product of $\mathbf{u}$ and $\mathbf{v}$ implies that $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$. Since $\sin\theta = \sqrt{1-\cos^2\theta}$, it follows that

$$A = ||\mathbf{u}||\ ||\mathbf{v}|| \sin\theta = ||\mathbf{u}||\ ||\mathbf{v}|| \sqrt{1 - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}\right)^2}$$

which simplifies to

$$A = \sqrt{||\mathbf{u}||^2\, ||\mathbf{v}||^2 - (\mathbf{u}\cdot\mathbf{v})^2} \tag{2}$$

Since $||\mathbf{u}||^2 = u_1^2 + u_2^2 + u_3^2$, $||\mathbf{v}||^2 = v_1^2 + v_2^2 + v_3^2$ and $(\mathbf{u}\cdot\mathbf{v})^2 = (u_1v_1 + u_2v_2 + u_3v_3)^2$, (2) becomes

$$A = \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2}$$

Expanding $\left(u_1^2+u_2^2+u_3^2\right)\left(v_1^2+v_2^2+v_3^2\right)$ gives

$$||\mathbf{u}||^2\,||\mathbf{v}||^2 = u_1^2v_1^2 \;+\; u_1^2v_2^2 \;+\; u_1^2v_3^2 \;+\; u_2^2v_1^2 \;+\; u_2^2v_2^2 \;+\; u_2^2v_3^2 \;+\; u_3^2v_1^2 \;+\; u_3^2v_2^2 \;+\; u_3^2v_3^2$$

To easily expand $(u_1v_1+u_2v_2+u_3v_3)^2$, we use the fact that $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$. The result is

$$(\mathbf{u}\cdot\mathbf{v})^2 = u_1^2v_1^2 \;+\; u_2^2v_2^2 \;+\; u_3^2v_3^2 \;+\; 2\,u_1v_1u_2v_2 \;+\; 2\,u_1v_1u_3v_3 \;+\; 2\,u_2v_2u_3v_3$$

Therefore $||\mathbf{u}||^2\,||\mathbf{v}||^2 - (\mathbf{u}\cdot\mathbf{v})^2$ may be written as

$$\left(u_1^2v_2^2 - \,2\,u_1v_1u_2v_2 + \,u_2^2v_1^2\right) + \left(u_1^2v_3^2 - \,2\,u_1v_1u_3v_3 + u_3^2v_1^2\right) + \left(u_2^2v_3^2 - \,2\,u_2v_2u_3v_3 + u_3^2v_2^2\right) \qquad (3)$$

The expressions in parentheses can be factored! When we do so, (3) becomes

$$(u_1v_2-u_2v_1)^2 + (u_1v_3-u_3v_1)^2 + (u_2v_3-u_3v_2)^2$$

Now the area of the parallelogram may be written as

$$A = \sqrt{(u_1v_2-u_2v_1)^2 + (u_1v_3-u_3v_1)^2 + (u_2v_3-u_3v_2)^2}$$

This is really the norm of some vector with components $\pm(u_1v_2-u_2v_1)$, $\pm(u_1v_3-u_3v_1)$ and $\pm(u_2v_3-u_3v_2)$. A convenient choice, (because, as you will soon find out, it gives a vector that is perpendicular to $\mathbf{u}$ and $\mathbf{v}$), is

$$(u_2v_3-u_3v_2)\,\mathbf{i} - (u_1v_3-u_3v_1)\,\mathbf{j} + (u_1v_2-u_2v_1)\,\mathbf{k}.$$

We call it the cross product of $\mathbf{u} =< u_1,u_2,u_3 >$ and $\mathbf{v} =< v_1,v_2,v_3 >$, and denote it by $\mathbf{u}\times\mathbf{v}$. We have therefore shown that the area of the parallelogram is $||\mathbf{u}\times\mathbf{v}||$. A formal definition of $\mathbf{u}\times\mathbf{v}$ is the following:

Definition 1 *The cross product of two given vectors* $\mathbf{u} =< u_1,u_2,u_3 >$ *and* $\mathbf{v} =< v_1,v_2,v_3 >$ *is denoted by* $\mathbf{u}\times\mathbf{v}$ *and is defined by*

$$\mathbf{u}\times\mathbf{v} = (u_2v_3-u_3v_2)\,\mathbf{i} - (u_1v_3-u_3v_1)\,\mathbf{j} + (u_1v_2-u_2v_1)\,\mathbf{k}$$

We also refer to $\mathbf{u}\times\mathbf{v}$ as the vector product of $\mathbf{u}$ and $\mathbf{v}$, because, unlike $\mathbf{u}\cdot\mathbf{v}$, it is a vector.
An easy way of remembering $\mathbf{u}\times\mathbf{v}$ is to view it as the "determinant"

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Examples:

1.

(a) $\mathbf{i}\times\mathbf{j}=\mathbf{k}$ (b) $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ (c) $\mathbf{k}\times\mathbf{i}=\mathbf{j}$ (d) $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$ (e) $\mathbf{k}\times\mathbf{j}=-\mathbf{i}$
(f) $\mathbf{i}\times\mathbf{k}=-\mathbf{j}$ (g) $\mathbf{i}\times\mathbf{i}=\mathbf{0}$ (h) $\mathbf{j}\times\mathbf{j}=\mathbf{0}$ (i) $\mathbf{k}\times\mathbf{k}=\mathbf{0}$

2. If $\mathbf{u}=(3,1,-2)=3\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{v}=(-1,0,4)=-\mathbf{i}+4\mathbf{k}$ then

$$\mathbf{u}\times\mathbf{v} = (4-0)\,\mathbf{i} - (12-2)\,\mathbf{j} + (0+1)\,\mathbf{k} = 4\mathbf{i} - 10\mathbf{j} + \mathbf{k}$$

Remark 2 *The area of the parallelogram OPQR, by formula (1), is* $||\mathbf{u}||\,||\mathbf{v}||\sin\theta$*. Therefore* $||\mathbf{u}\times\mathbf{v}|| = ||\mathbf{u}||\,||\mathbf{v}||\sin\theta$*, hence* $\mathbf{u}\times\mathbf{v}$ *is a vector with magnitude* $||\mathbf{u}||\,||\mathbf{v}||\sin\theta$*. Its direction is determined by the following right-hand rule: Imagine rotating* $\mathbf{u}$ *about the origin, towards* $\mathbf{v}$*. Now imagine turning an ordinary screw in the same way. The direction of* $\mathbf{u}\times\mathbf{v}$ *is given by the direction in which the screw moves. This is in perfect agreement with results* $\mathbf{i}\times\mathbf{j}=\mathbf{k}$*,* $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$*, etc in the above example*

A number of useful properties of the cross product are given by the following theorem

Theorem 3 *Let* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be given vectors and* λ *be a real number. Then*

1. $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$. *Thus the cross product operation is NOT commutative.*
2. $\mathbf{u} \times (\lambda \mathbf{v}) = \boldsymbol{\lambda} \mathbf{u} \times \mathbf{v}$.
3. $\mathbf{u} \times \mathbf{0} = \mathbf{0}$. *Here,* $\mathbf{0}$ *is the zero vector* $< 0, 0, 0 >$.
4. $\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$ *and it is also orthogonal to* $\mathbf{v}$.
5. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$. *In other words, the cross product operation is distributive with respect to addition.*
6. *If* $\mathbf{u} =< u_1, u_2, u_3 >$, $\mathbf{v} =< v_1, v_2, v_3 >$ *and* $\mathbf{w} =< w_1, w_2, w_3 >$ *then*

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

7. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
8. *If* $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ *then one of the two vectors is a scalar multiple of the other. Conversely, if one of* $\mathbf{u}$,$\mathbf{v}$ *is a scalar multiple of the other then* $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

To prove these claims, let $\mathbf{u} =< u_1, u_2, u_3 >$, $\mathbf{v} =< v_1, v_2, v_3 >$ and $\mathbf{w} =< w_1, w_2, w_3 >$.

1. By definition,

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \quad \text{and} \quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

The first one expands as $(v_2 u_3 - u_2 v_3)\,\mathbf{i} - (v_1 u_3 - u_{13} v_3)\,\mathbf{j} + (u_2 v_1 - u_1 v_2)\,\mathbf{k}$ and the second one as $(u_2 v_3 - v_2 u_3)\,\mathbf{i} - (u_{13} v_3 - v_1 u_3)\,\mathbf{j} + (u_{13} v_2 - u_2 v_1)\,\mathbf{k}$. Clearly, the second one is the negative of the first one, therefore $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$.

2. Since $\lambda v =< \lambda v_1, \lambda v_2, \lambda v_3 >$,

$$\mathbf{u} \times (\lambda \mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ \lambda v_1 & \lambda v_2 & \lambda v_3 \end{vmatrix} = (\lambda u_2 v_3 - \lambda v_2 u_3)\,\mathbf{i} - (\lambda u_1 v_3 - \lambda v_1 u_3)\,\mathbf{j} + (\lambda u_1 v_2 - \lambda u_2 v_1)\,\mathbf{k}$$

The right hand side may be written as $\lambda[(u_2 v_3 - v_2 u_3)\,\mathbf{i} - (u_1 v_3 - v_1 u_3)\,\mathbf{j} + (u_1 v_2 - u_2 v_1)\,\mathbf{k}] = \boldsymbol{\lambda} \mathbf{u} \times \mathbf{v}$

3.

$$\mathbf{u} \times \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ 0 & 0 & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

4. To show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$, it suffices to show that their dot product is zero.

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= u_1 (u_2 v_3 - v_2 u_3) - u_2 (u_1 v_3 - v_1 u_3) + u_3 (u_1 v_2 - u_2 v_1) \\
&= u_1 u_2 v_3 - u_1 v_2 u_3 - u_1 u_2 v_3 + u_2 v_1 u_3 + u_1 v_2 u_3 - u_2 v_1 u_3 = 0,
\end{aligned}$$

That $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{v}$ is verified in the same way.

5. By definition, $\mathbf{v} + \mathbf{w} =< v_1 + w_1, v_2 + w_2, v_3 + w_3 >$. Therefore

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix}$$

Expanding gives

$$[u_2 (v_3 + w_3) - u_3 (v_2 + w_2)]\, \mathbf{i} - [u_1 (v_3 + w_3) - u_3 (v_1 + w_1)]\, \mathbf{j} + [u_1 (v_2 + w_2) - u_2 (v_1 + w_1)]\, \mathbf{k}.$$

We may rearrange this as

$$\begin{aligned}
&[(u_2 v_3 - v_2 u_3)\, \mathbf{i} - (u_1 v_3 - v_1 u_3)\, \mathbf{j} + (u_1 v_2 - u_2 v_1)\, \mathbf{k}] \\
&+ [(u_2 w_3 - w_2 u_3)\, \mathbf{i} - (u_1 w_3 - w_1 u_3)\, \mathbf{j} + (u_1 w_2 - u_2 w_1)\, \mathbf{k}]
\end{aligned}$$

which is equal to $\quad \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

6. By definition,

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\, \mathbf{i} - (v_1 w_3 - v_3 w_1)\, \mathbf{j} + (v_1 w_2 - w_1 v_2)\, \mathbf{k}$$

It follows that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_1(v_2 w_3 - v_3 w_2) - u_2(v_1 w_3 + v_3 w_1) + u_3(v_1 w_2 - w_1 v_2)$. Expanding

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

gives the same result.

7.

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= u_1 v_2 w_3 - u_1 v_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 w_1 v_2 \\
&= (u_2 v_3 - u_3 v_2)\, w_1 - (u_1 v_3 - u_3 v_1)\, w_2 + (u_1 v_2 - u_2 v_1)\, w_3 = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}
\end{aligned}$$

8. Suppose $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. We may assume that both vectors are nonzero because if one of them is zero then it is a scalar multiple of the other vector. For example, if $\mathbf{u}$ is zero then it is a scalar multiple of $\mathbf{v}$ since we may write

$$\mathbf{u} = \mathbf{0} = 0\mathbf{v}$$

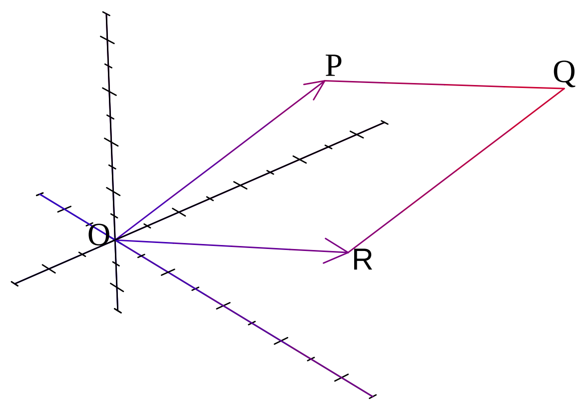
Let θ be the angle between the two nonzero vectors. Since $\mathbf{u} \times \mathbf{v} = ||\mathbf{u}||\; ||\mathbf{v}|| \sin\theta = 0$, and the product $||\mathbf{u}||\; ||\mathbf{v}||$ is nonzero, it follows that $\sin\theta = 0$. This in turn implies that $\theta = 0$ or $180°$, therefore $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. Conversely, if say $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$, then $\mathbf{u} = \lambda\mathbf{v}$ where λ is a real number, and so

$$\mathbf{u} \times \mathbf{v} = \lambda\mathbf{v} \times \mathbf{v} = \lambda \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lambda v_1 & \lambda v_2 & \lambda v_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

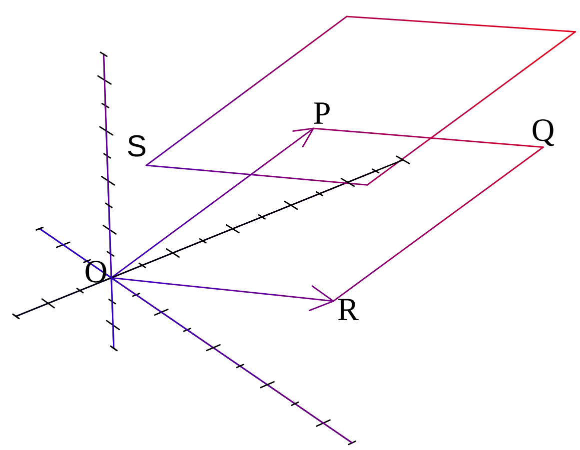
Volume of a parallelepiped

We will run into volumes of parallelepipeds when handling volume integrals. Here is how to construct a parallelepiped:

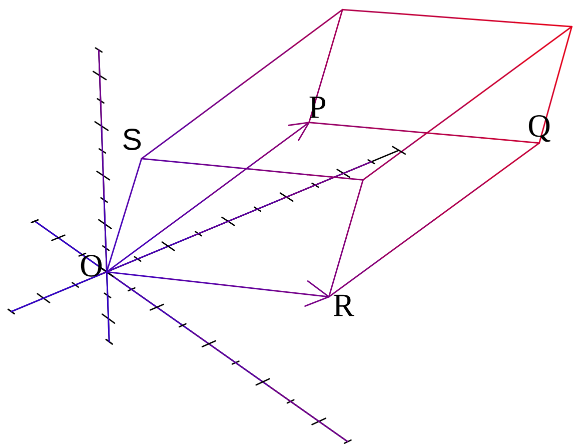
1. Start with a parallelogram $OPQR$, (an example is shown below).



2. Translate a copy of $OPQR$ by a vector $\mathbf{u} = \vec{OS}$



3. Now join the corresponding corners of the two parallelograms to get a solid, called a parallelepiped.



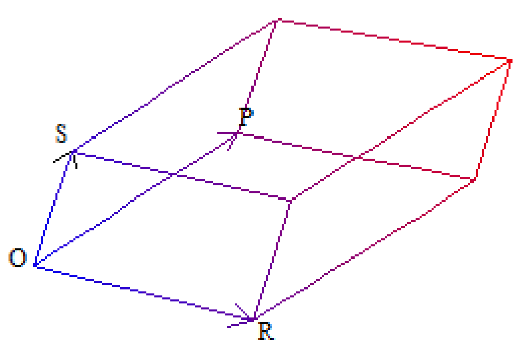
To calculate its volume, simply multiply the area of the parallelogram $OPQR$ by the vertical distance between the two parallelograms. The vertical distance from the point S to the parallelogram $OPQR$ is equal to the dot product of $\vec{OS}$ and a unit vector perpendicular to the parallelogram. (This is a result of a problem solved under dot products.) Since $\left(\vec{OR}\right) \times \left(\vec{OP}\right)$ is a vector perpendicular to the parallelogram, the vertical distance between them, is

$$\left| \vec{OS} \cdot \frac{\left(\vec{OR}\right) \times \left(\vec{OP}\right)}{\left\| \left(\vec{OR}\right) \times \left(\vec{OP}\right) \right\|} \right| = \left| \frac{\vec{OS} \cdot \left(\vec{OR}\right) \times \left(\vec{OP}\right)}{\left\| \left(\vec{OR}\right) \times \left(\vec{OP}\right) \right\|} \right| = \frac{\left| \vec{OS} \cdot \left(\vec{OR}\right) \times \left(\vec{OP}\right) \right|}{\left\| \left(\vec{OR}\right) \times \left(\vec{OP}\right) \right\|}$$

We take the absolute value because the distance must be non-negative. But the area of the parallelogram is $\left\| \left(\vec{OR}\right) \times \left(\vec{OP}\right) \right\|$, therefore the volume of the parallelepiped is

$$\frac{\left| \vec{OS} \cdot \left(\vec{OR}\right) \times \left(\vec{OP}\right) \right| \left\| \left(\vec{OR}\right) \times \left(\vec{OP}\right) \right\|}{\left\| \left(\vec{OR}\right) \times \left(\vec{OP}\right) \right\|} = \left| \vec{OS} \cdot \left(\vec{OR}\right) \times \left(\vec{OP}\right) \right|$$

In general, let $\mathbf{u} = \vec{OP}$, $\mathbf{v} = \vec{OR}$ and $\mathbf{w} = \vec{OS}$ be vectors that are not in the same plane.



They generate a parallelepiped with volume $|\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}|$. Using a property we derived above, it follows that if $\mathbf{u} =< u_1, u_2, u_3 >$, $\mathbf{v} =< v_1, v_2, v_3 >$ and $\mathbf{w} =< w_1, w_2, w_3 >$ then the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is the absolute value of

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Exercise 4

1. *Compute the cross product* $\mathbf{u} \times \mathbf{v}$ *given that*

   (a) $\mathbf{u} = < 2, 0, -1 >$ *and* $\mathbf{v} =< -1, 0, 2 >$.

   $\mathbf{u} = < 4, 4, -1 >$ *and* $\mathbf{v} =< -1, 1, -3 >$.

   $\mathbf{u} = < a, 2, -1 >$ *and* $\mathbf{v} =< -a, 0, -3 >$.

   $\mathbf{u} = < a, a, 1 >$ *and* $\mathbf{v} =< a, a, a >$.

2. *Use the cross product of* $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ *and* $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$ *to calculate the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

3. *Calculate the area of the parallelogram determined by the two vectors* $< 2, -1, 4 >$ *and* $< 3, 3, 5 >$.

4. *Calculate the area of the triangle with vertices at* $(1, 1, 4)$, $(2, 5, -3)$ *and* $(-1, 2, -4)$.

5. *Calculate the volume of the parallelepiped generated by the vectors* $< 4, 5, 0 >$, $< 3, 7, 0 >$ *and* $< 1, 2, 5 >$.

6. *True or False? If* $\mathbf{u} \times \mathbf{v} = \mathbf{u} \cdot \mathbf{v} = \mathbf{0}$ *then* $\mathbf{u} = \mathbf{0}$ *or* $\mathbf{v} = \mathbf{0}$. *If true, prove it. If false, give a counterexample.*

7. *Use appropriate properties of determinants to show that* $|\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}| = |\mathbf{v} \cdot \mathbf{w} \times \mathbf{u}| = |\mathbf{w} \cdot \mathbf{u} \times \mathbf{v}|$