Properties of Limits

In practice, it is not necessary to use the definition of a limit to determine limits of the familiar functions. Instead, we determine the limits of a few elementary functions and derive some useful properties of limits. Then given a function f, we write it as a sum or product or quotient or composition of the elementary functions and use the properties of limits to determine its limit. The first set of properties of limits is given by the following theorem:

Theorem 1 *Let f and g be given real-valued functions of two variables x and y. Let (c, d) be a given point in the plane and L, M be given real numbers. If $\lim_{(x,y)\to(c,d)} f(x, y) = L$ and $\lim_{(x,y)\to(c,d)} g(x, y) = M$ then:*

1. $\lim_{(x,y)\to(c,d)} [f(x, y) + g(x, y)] = L + M$ *and* $\lim_{(x,y)\to(c,d)} [f(x, y) - g(x, y)] = L - M$
2. $\lim_{(x,y)\to(c,d)} \lambda f(x, y) = \lambda L$ *for every real number* λ.
3. $\lim_{(x,y)\to(c,d)} [f(x, y)g(x, y)] = LM$
4. $\lim_{(x,y)\to(c,d)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{L}{M}$ *provided* $M \neq 0$

Proof. *Let $\varepsilon > 0$ be given:*

1. *We have to show that there is a $\delta > 0$ with the property that if $0 < ||(x - c, y - d)|| < \delta$ then*

$$|f(x, y) + g(x, y) - (L + M)| < \varepsilon$$

Note that $f(x, y) + g(x, y) - (L + M)$ may be re-arranged as

$$(f(x, y) - L) + (g(x, y) - M)$$

and the triangle inequality implies that

$$|(f(x, y) - L) + (g(x, y) - M)| \leq |f(x, y) - L| + |g(x, y) - M|$$

Therefore it suffices to show that there is a $\delta > 0$ such that $|f(x, y) - L| < \frac{1}{2}\varepsilon$ and $|g(x, y) - M| < \frac{1}{2}\varepsilon$ whenever $0 < ||(x - c, y - d)|| < \delta$. This is where the fact that f has limit L and g has limit M as (x, y) approaches (c, d) is used. It guarantees that there is $\delta_1 > 0$ such that

$$|f(x, y) - L| < \tfrac{1}{2}\varepsilon$$

whenever $0 < ||(x - c, y - d)|| < \delta_1$ and that there is $\delta_2 > 0$ such that

$$|g(x, y) - L| < \tfrac{1}{2}\varepsilon$$

whenever $0 < ||(x - c, y - d)|| < \delta_2$. Take δ to be the smaller of the two numbers δ_1 and δ_2. That is, take $\delta = \min\{\delta_1, \delta_2\}$. Then $|f(x, y) - L| < \frac{1}{2}\varepsilon$ and $|g(x, y) - M| < \frac{1}{2}\varepsilon$ whenever $0 < ||(x - c, y - d)|| < \delta$, which implies that

$$|f(x, y) + g(x, y) - (L + M)| \leq |f(x, y) - L| + |g(x, y) - M| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

In other words, if $0 < ||(x - c, y - d)|| < \delta$ then $|f(x, y) + g(x, y) - (L + M)| < \varepsilon$, which proves that

$$\lim_{(x,y)\to(c,d)} [f(x, y) + g(x, y)] = L + M.$$

The proof that $\lim_{(x,y)\to(c,d)} [f(x, y) - g(x, y)] = L - M$ is similar.

2. *If $\lambda = 0$ then $\lambda f(x,y) = 0$ for all (x,y), hence $\lim_{(x,y)\to(c,d)} \lambda f(x,y) = 0 = \lambda L$. Therefore we may assume that $\lambda \neq 0$. We have to show that there is $\delta > 0$ such that*

$$|\lambda f(x,y) - \lambda L| < \varepsilon$$

whenever $0 < ||(x-c, y-d)|| < \delta$. Clearly, $|\lambda f(x,y) - \lambda L|$ may be written as $|\lambda| \left(|f(x,y) - L|\right)$. Therefore it suffices to show that there is $\delta > 0$ such that

$$|f(x,y) - L| < \frac{\varepsilon}{|\lambda|}$$

whenever $0 < ||(x-c, y-d)|| < \delta$. But this is true because f has limit L as (x,y) approaches (c,d). In other words, given any positive number ε, we can find $\delta > 0$ such that $|f(x,y) - L| < \dfrac{\varepsilon}{|\lambda|}$ whenever $0 < ||(x-c, y-d)|| < \delta$, which implies that

$$|\lambda f(x,y) - \lambda L| < \frac{|\lambda|\,\varepsilon}{|\lambda|} = \varepsilon.$$

The proofs of parts 3 and 4 are exercises.

■

Example 2 *We showed that $f(x,y) = x$ has limit c and that $g(x,y) = y$ has limit d as (x,y) approaches (c,d). Let k and q be constants. Then by property 2, $u(x,y) = kx$ has limit kc and $v(x,y) = qy$ has limit qd as (x,y) approaches (c,d). If we combine this with property 1 then we easily conclude that $w(x,y) = kx + qy$ has limit $kc + qd$ as (x,y) approaches any given point (c,d).*

Example 3 *Since $f(x,y) = x$ has limit c and that $g(x,y) = y$ has limit d as (x,y) approaches (c,d), property 3, implies that $r(x,y) = xy$ has limit cd, $s(x,y) = x^2$ has limit c^2 and $t(x,y) = y^2$ has limit d^2 as (x,y) approaches (c,d). In general, if k and n are nonnegative integers then $h(x,y) = x^k y^n$ has limit $c^k d^n$ as (x,y) approaches any given point (c,d).*

Exercise 4

1. *Use a right triangle in a plane to verify that if (x,y) and (c,d) are points in a plane then*

$$|x-c| \leq ||(x-c, y-d)|| \qquad \text{and} \qquad |y-d| \leq ||(x-c, y-d)||\,.$$

Under what condition(s) is $|x-c| = ||(x-c, y-d)||$?

2. *A function $f(x,y)$ has a limit as (x,y) approaches a given point (c,d) if every pair (x,y) near (c,d) gives a value $f(x,y)$ close to some* ***fixed number****. It is that fixed number that we call the limit of f. Consider the function $f(x,y) = \dfrac{x^2 - 2y^2}{x^2 + y^2}$, $(x,y) \neq (0,0)$. Take $(c,d) = (0,0)$. It turns out that all the pairs (x,y) on the straight line $y = x$ through the origin give the same value $f(x,x) = \dfrac{-x^2}{2x^2} = -\frac{1}{2}$. On the other hand, all the pairs $(x,0)$ on the x-axis give another value, namely $f(x,0) = \dfrac{x^2}{x^2} = 1$. Therefore it is not true that all the pairs (x,y) that are close to $(0,0)$ give values $f(x,y)$ close to a fixed number (since there are pairs on the line $y = x$ that give value $-\frac{1}{2}$ and other pairs on the x-axis that give value 1). Therefore this function has no limit as (x,y) approaches $(0,0)$.*

   (a) *Consider the function $g(x,y) = \dfrac{x^2 + xy}{x^2 + y^2}$, $(x,y) \neq (0,0)$. Determine its value on the line $y = x$ and its value on the x-axis then deduce that it has no limit as (x,y) approaches $(0,0)$.*

   (b) *Consider the function $f(x,y) = \dfrac{\sqrt{x^4 + y^4}}{x^2 + y^2}$, $(x,y) \neq (0,0)$. Determine its value on the line $y = x$ and its value on the line $x = 2x$ then deduce that it has no limit as (x,y) approaches $(0,0)$.*

(c) *Consider the function* $h(x,y) = \dfrac{xy\sqrt{x^2+y^2+1}}{x^2+y^2}$, $(x,y) \neq (0,0)$. *Show that if* $x \neq 0$ *then its value at a point* (x,x) *is* $\dfrac{\sqrt{2x^2+1}}{2}$. *Does* h *have a limit as* (x,y) *approaches* $(0,0)$? *Defend your answer.*

3. *Let* k *be a constant real number. Use the definition of a limit to show that the constant function* $f(x,y) = k$ *has limit* k *as* (x,y) *approaches any given point* (c,d).

4. *Use the fact that* $f(x,y) = x$ *has limit* c, $g(x,y) = y$ *has limit* d *as* (x,y) *approaches* (c,d) *and appropriate properties of limits to show that:*

   (a) $h(x,y) = 3x^2y - 4y^2$ *has limit* $3c^2d - 4d^2$ *as* (x,y) *approaches any given point* (c,d).

   (b) $w(x,y) = \dfrac{x^3-3y^2+5}{x^2+y^2+1}$ *has limit* 5 *as* (x,y) *approaches* $(0,0)$.

   (c) $v(x,y) = \dfrac{x-y+1}{x^2+y^2}$ *has limit* $\dfrac{c-d+1}{c^2+d^2}$ *as* (x,y) *approaches any given point* (c,d) *different from* $(0,0)$.

   (d) *If* $p(x,y)$ *is a polynomial in* x *and* y *then* $\lim\limits_{(x,y)\to(c,d)} p(x,y) = p(c,d)$.

   (e) *If* $p(x,t)$ *and* $q(x,y)$ *are polynomials in* x *and* y *and if* $q(c,d) \neq 0$ *then* $\lim\limits_{(x,y)\to(c,d)} \dfrac{p(x,y)}{q(x,y)} = \dfrac{p(c,d)}{q(c,d)}$.

5. *In this exercise, you prove part 3 of Theorem 1. Thus* f *and* g *are given real-valued functions of two variables* x *and* y, (c,d) *is a given point in the plane* L *and* M *are given real numbers,* $\lim\limits_{(x,y)\to(c,d)} f(x,y) = L$ *and* $\lim\limits_{(x,y)\to(c,d)} g(x,y) = M$. *You have to prove that* $\lim\limits_{(x,y)\to(c,d)} f(x,y)g(x,y) = LM$. *Fill in the missing details in the following outline:*

   Let $\varepsilon > 0$ *be given. We have to show that there is* $\delta > 0$ *such that*

   $$|f(x,y)g(x,y) - LM| < \varepsilon$$

   whenever $0 < ||(x-c, y-d)|| < \delta$. *A trick that brings* $|f(x,y) - L|$ *and* $|g(x,y) - M|$ *into the above inequality is to add* $-lg(x,y) + lg(x,y)$, *(which is zero), to* $f(x,y)g(x,y) - LM$ *and rearrange the terms. Do so then show that*

   $$|f(x,y)g(x,y) - LM| \leq |g(x,y)|\,|f(x,y) - L| + |L|\,|g(x,y) - M|$$

   Now we see that it suffices to show that there is a $\delta > 0$ *such that*

   $$|g(x,y)|\,|f(x,y) - L| < \tfrac{1}{2}\varepsilon \qquad \text{and} \qquad |L|\,|g(x,y) - M| < \tfrac{1}{2}\varepsilon$$

   whenever $0 < ||(x-c, y-d)|| < \delta$. *Unfortunately, because* $|g(x,y)|$ *is not a constant, it is hard to find a* $\delta > 0$ *satisfying* $|g(x,y)|\,|f(x,y) - L| < \frac{1}{2}\varepsilon$. *One way of getting around this problem is to replace* $|g(x,y)|$ *with a constant that is bigger than* $|g(x,y)|$. *Since* $\lim\limits_{(x,y)\to(c,d)} g(x,y) = M$, *there is a* $\delta_1 > 0$ *such that*

   $$|g(x,y) - M| < 2$$

   whenever $0 < ||(x-c, y-d)|| < \delta_1$. *(There is nothing special about the number 2. Any positive constant will work.) Now write* $|g(x,y)|$ *as* $|[g(x,y) - M] + M|$ *then use the triangle inequality, to conclude that if* $0 < ||(x-c, y-d)|| < \delta_1$ *then*

   $$|g(x,y)| = |[g(x,y) - M] + M| \leq |g(x,y) - M| + |M| < 2 + |M|.$$

   Use this to show that if $0 < ||(x-c, y-d)|| < \delta_1$ *then*

   $$|f(x,y)g(x,y) - LM| \leq (2 + |M|)\,|f(x,y) - L| + |L|\,|g(x,y) - M|.$$

Since f has limit L as (x,y) approaches (c,d), there is $\delta_2 > 0$ such that

$$|f(x,y) - L| < \frac{\varepsilon}{2\,(2 + |M|)}$$

whenever $0 < ||(x-c, y-d)|| < \delta_2$. Since g has limit M as (x,y) approaches (c,d), there is $\delta_3 > 0$ such that

$$|g(x,y) - M| < \frac{\varepsilon}{2\,(1 + |L|)}$$

whenever $0 < ||(x-c, y-d)|| < \delta_3$. (The choice $\dfrac{\varepsilon}{2\,(1+|L|)}$ instead of $\dfrac{\varepsilon}{2\,|L|}$ guarantees that we do not divide by 0 if L should happen to be 0.) Now show that if we choose δ to be the smallest of the three numbers δ_1, δ_2 and δ_3 then

$$|f(x,y)g(x,y) - LM| < \varepsilon$$

whenever $0 < ||(x-c, y-d)|| < \delta$ and complete the proof of the theorem.

6. *In this exercise, you prove part 4 of Theorem 1. Thus f and g are given real-valued functions of two variables x and y, (c,d) is a given point in the plane L and M are given real numbers with $M \neq 0$, $\lim\limits_{(x,y)\to(c,d)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(c,d)} g(x,y) = M$. You have to prove that $\lim\limits_{(x,y)\to(c,d)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$. Fill in the missing details in the following outline:*

   Let ε be a given positive number. We have to show that there is $\delta > 0$ such that

$$\left| \frac{f(x,y)}{g\,(x,y)} - \frac{L}{M} \right| < \varepsilon$$

   whenever $0 < ||(x-c, y-d)|| < \delta$. Begin by showing that

$$\left| \frac{f(x,y)}{g\,(x,y)} - \frac{L}{M} \right| = \left| \frac{Mf\,(x,y) - Lg\,(x,y)}{Mg\,(x,y)} \right|$$

   A trick that brings $|f(x,y) - L|$ and $|g(x,y) - M|$ into the above inequality is to add $-LM + LM$ (which is zero), to the numerator and rearrange the terms. Do so then show that

$$\left| \frac{f(x,y)}{g\,(x,y)} - \frac{L}{M} \right| \leq \frac{|f(x,y) - L|}{|g(x,y)|} + \frac{|L|\,|g(x,y) - M|}{|M|\,|g(x,y)|}$$

   Now we see that it suffices to show that there is a $\delta > 0$ such that

$$\frac{|f(x,y) - L|}{|g(x,y)|} < \tfrac{1}{2}\varepsilon \qquad \text{and} \qquad \frac{|L|\,|g(x,y) - M|}{|M|\,|g(x,y)|} < \tfrac{1}{2}\varepsilon$$

   whenever $0 < ||(x-c, y-d)|| < \delta$. Unfortunately, because $|g(x,y)|$ is not a constant, such a δ is hard to find. To get around this problem, replace $|g(x,y)|$ with a positive constant that is smaller than $|g(x,y)|$ on some punctured disc centred at (c,d) as follows: Since $\lim\limits_{(x,y)\to(c,d)} g(x,y) = M$, and $\frac{1}{2}|M|$ is positive, there is a $\delta_1 > 0$ such that

$$|g(x,y) - M| < \tfrac{1}{2}|M|$$

   whenever $0 < ||(x-c, y-d)|| < \delta_1$. Now use the triangle inequality in the form $|a| - |b| \leq |a-b|$ to show that $\frac{1}{2}|M| < |g(x,y)|$ whenever $0 < ||(x-c, y-d)|| < \delta_1$, then show that

$$\left| \frac{f(x,y)}{g\,(x,y)} - \frac{L}{M} \right| \leq \frac{2\,|f(x,y) - L|}{|M|} + \frac{2\,|L|\,|g(x,y) - M|}{|M|^2}$$

   Since $\lim\limits_{(x,y)\to(c,d)} f(x,y) = L$ there is a $\delta_2 > 0$ such that $|f(x,y) - L| < \dfrac{|M|}{4}\varepsilon$ whenever $0 < ||(x-c, y-d)|| < \delta_2$. Continue and complete the proof.

Continuous Functions

A function f of two variables is continuous at a point (c,d) if pairs (x,y) close to (c,d) give values $f(x,y)$ close to $f(c,d)$, (the value of f at (c,d)). Of course f must be defined at (c,d) and at all points in some disc centred at (c,d).

A precise way of saying that points (x,y) close to (c,d) give values $f(x,y)$ is close to $f(c,d)$ is that $f(x,y)$ has limit $f(c,d)$ as (x,y) approaches (c,d), hence the following definition::

Definition 5 *A function f of two variables is continuous at a point (c,d) if:*

1. *(c,d) is in the domain of f and the function is defined in some disc centred at (c,d).*
2. *Given any positive number ε, it is possible to find $\delta > 0$ such that $|f(x,y) - f(c,d)| < \varepsilon$ whenever $||(x-c, y-d)|| < \delta$.*

We showed that if f is a polynomial in x and $y$and (c,d) is any point in the plane then $\lim_{(x,y)\to(c,d)} f(x,y) = f(c,d)$, therefore polynomial functions are continuous at any point in the plane. The rational functions $\dfrac{g(x,y)}{h(x,y)}$ are continuous at every point (c,d) where $h(c,d) \neq 0$.

Using properties of limits, one shows that if f and g are continuous at a point (c,d) then:

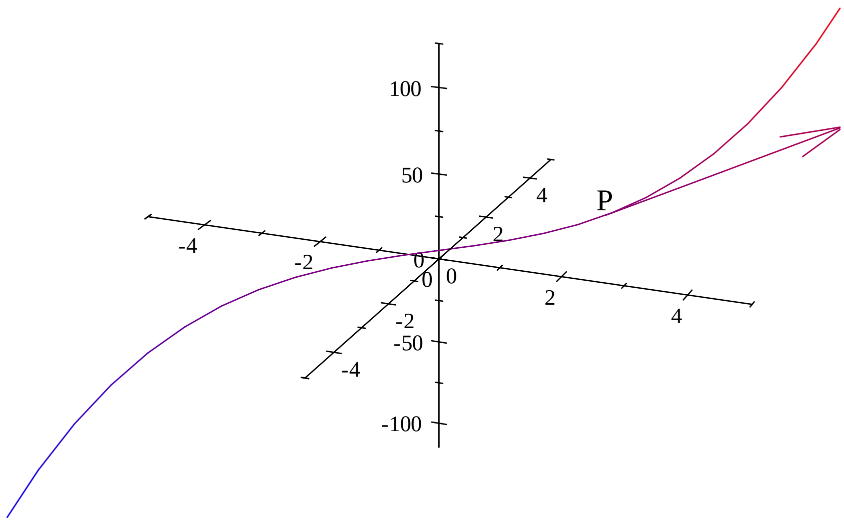
1. Their sum $f+g$ and difference $f-g$ is also continuous at (c,d). (This is because, if $\lim_{(x,y)\to(c,d)} f(x,y) = f(c,d)$ and $\lim_{(x,y)\to(c,d)} g(x,y) = g(c,d)$ then $\lim_{(x,y)\to(c,d)} [f(x,y) \pm g(x,y] = f(c,d) \pm g(c,d)$.)
2. kf is continuous at (c,d) for any constant k. (This is because, if $\lim_{(x,y)\to(c,d)} f(x,y) = f(c,d)$ then $\lim_{(x,y)\to(c,d)} kf(x,y) = kf(c,d)$.)
3. Their product gf is continuous at (c,d). (This is because, if $\lim_{(x,y)\to(c,d)} f(x,y) = f(c,d)$ and $\lim_{(x,y)\to(c,d)} g(x,y) = g(c,d)$ then $\lim_{(x,y)\to(c,d)} [f(x,y)g(x,y] = f(c,d)g(c,d)$.)
4. The quotient $\dfrac{f(x,y)}{g(x,y)}$ is continuous at (c,d) provided $g(c,d) \neq 0$. (This is because, if $\lim_{(x,y)\to(c,d)} f(x,y) = f(c,d)$, $\lim_{(x,y)\to(c,d)} g(x,y) = g(c,d)$ and $g(c,d) \neq$ then $\lim_{(x,y)\to(c,d)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{f(c,d)}{g(c,d)}$.)

Exercise 6

1. *Let $f(x,y) = x^2 + xy - 2y^2$. Determine a $\delta > 0$ such that $|f(x,y) - f(3,-1)| < 0.01$ whenever $||(x,y) - (3,-1)|| < \delta$.*
2. *Complete the following sentence: A function $f(x,y)$ is NOT continuous at a point (c,d) if ...*
3. *Let $f(x,y) = \dfrac{x+y}{\sqrt{x^2+y^2}}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 1$.*
   - *(a) Use the inequalities $|x| \leq ||(x,y)||$ and $|y| < ||(x,y)||$ to show that $|f(x,y)| \leq 2$ for all $(x,y) \in \mathbb{R}^2$. (Thus the values of f do not get arbitrarily large as you would expect from a function whose denominator approaches 0 as $(x,y) \to (0,0)$.)*
   - *(b) Prove that f is NOT continuous at $(0,0)$.*

A Tangent Vector to a Curve in Space

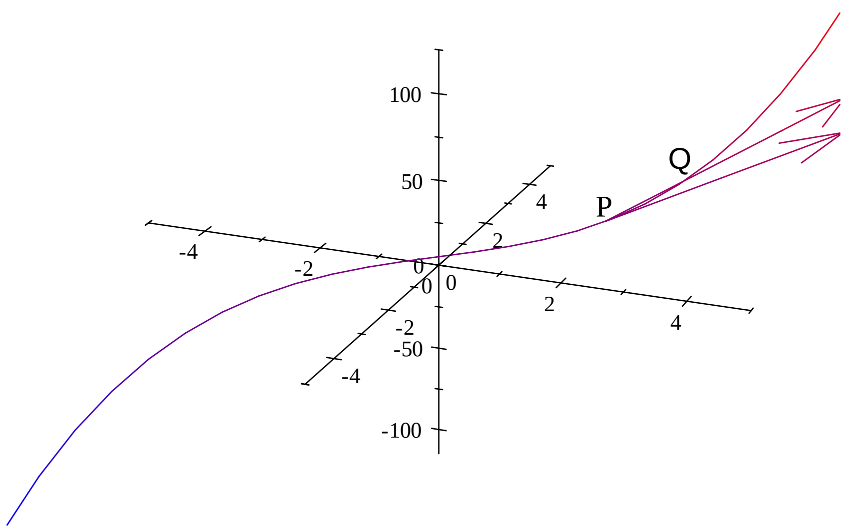
Let $\mathbf{c}(t) = (f(t), g(t), h(t))$ be a curve in space and $P\,(f(t_0), g(t_0), h(t_0))$ be a point on the curve. A tangent vector to $\mathbf{c}$ at P is a vector that lies flat on the curve at P. Its length is immaterial.



A tangent vector at P

To get an approximation to a tangent vector, we take a vector $\overrightarrow{PQ}$ where $Q\,(f(t), g(t), h(t))$ is a point on the curve that is close to P. Actually, when $Q\,(f(t), g(t), h(t))$ is close to P then $\overrightarrow{PQ}$ is a very "tiny" vector, (i.e. it has a very small norm). Since the length of a tangent vector may be immaterial, a better choice is

$$\tfrac{1}{(t-t_0)}\overrightarrow{PQ} \tag{1}$$



An approximate tangent vector $\tfrac{1}{(t-t_0)}\overrightarrow{PQ}$

The above diagram suggests that the approximation gets better as t approaches t_0, therefore it is reasonable to expect the limit of (1), (provided it exists), as t approaches t_0 to be a tangent vector at P. Since

$$\tfrac{1}{(t-t_0)}\overrightarrow{PQ} = < \frac{f(t)-f(t_0)}{t-t_0}, \frac{g(t)-g(t_0)}{t-t_0}, \frac{h(t)-h(t_0)}{t-t_0} > ,$$

the limit is guaranteed to exist provided f, g and h are differentiable functions of t. In such a case, a tangent vector at P is the vector $< f'(t_0), g'(t_0), h'(t_0) >$. This suggests the following definition:

Definition 7 *Let f, g and h be differentiable functions of one variable t and $\mathbf{c}(t) = (f(t), g(t), h(t))$ be a curve in space. Then $< f'(t_0), g'(t_0), h'(t_0) >$ is called a tangent vector to the curve at the point $(f(t_0), g(t_0), h(t_0))$.*

Exercise 8

1. *Determine a tangent vector to the given curve $\mathbf{c}(t)$ at the given point P.*

   (a) $\mathbf{c}(t) = \left(t\cos t, t\sin t, t^2\right)$ *at* $\mathbf{c}(\frac{\pi}{4})$.

   (b) $\mathbf{c}(t) = \left(e^t, t, e^{2t}\right)$ *at* $\mathbf{c}(0)$.

2. *Let $f(x, y) = x + y + x\sin y$ be the function in Example* **??**. *Consider its "$x = 2$ section". Determine a tangent to the curve at $(2, 0)$.*

3. *Let $f(x, y) = x + x^2 \sin y$ be the function in Exercise* **??**. *Consider its "$y = -\frac{\pi}{2}$ section". Determine a tangent to the curve at $\left(0, -\frac{\pi}{2}\right)$.*

4. *Let $f(x, y)$ be a given function of two variables and (a, b) be a point in its domain. Denote the $x = a$ section of f by $u(y)$. Thus $u(y) = f(a, y)$. Show that $< 0, 1, u'(b) >$ is a tangent vector to the x-section at the point (a, b).*

5. *Let $f(x, y)$ be a given function of two variables and (a, b) be a point in its domain. Denote the "$y = b$ section of f" by $v(x)$. Thus $v(x) = f(x, b)$. Show that $< 1, 0, v'(b) >$ is a tangent vector to the y-section at the point (a, b).*