Factoring

Factoring is "undoing" multiplication. When we factor a whole number, we reverse multiplication to write the number as a product of smaller whole numbers called its factors or divisors. Likewise, when we factor an algebraic expression, we reverse multiplication to write the expression as a product of simpler terms, also called its factors or divisors. We will use the idea of factoring whole numbers to motivate the process of factoring algebraic expressions. To this end, assume that we are given a big number. Factoring it amounts to writing it as a product of "prime numbers". As you probably already know, a prime number is an integer bigger than 1 that cannot be written as a product of smaller whole numbers. The only numbers that divide a prime number are the number itself and 1. Clearly, the first few prime numbers in increasing order are 2, 3, 5, 7, 11, 13. . . . It can be proved that every whole number may be written in exactly one way as a product of prime numbers, (this is called the "Fundamental Theorem of Arithmetic"), and the examples below show a standard way of doing that. It is essentially backwards division starting with the smallest prime number that divides the given number.

Example 1 *To write 350 as a product of prime factors we note that 2 divides 350, therefore we divide 350 by 2 to get* $350 = 2 \times 175$*. Now we look for the smallest prime that divides 175. It is 5, therefore we divide 175 by 5 to get* $175 = 5 \times 35$*. The next step should be clear; we look for the smallest prime that divided 35 and the answer is 5, therefore* $35 = 5 \times 7$*. Since 7 is prime, we stop. Now we see that*

$$350 = 2 \times 5 \times 5 \times 7$$

which is a product of prime factors. We may carry out these steps more systematically as follows:

$$\begin{aligned} 350 &= 2 \times 175 \\ &= 2 \times 5 \times 35 \\ &= 2 \times 5 \times 5 \times 7 \end{aligned}$$

Example 2 *To write 1080 as a product of prime factors:*

$$\begin{aligned} 1080 &= 2 \times 540 \\ &= 2 \times 2 \times 270 \\ &= 2 \times 2 \times 2 \times 135 \\ &= 2 \times 2 \times 2 \times 3 \times 45 \\ &= 2 \times 2 \times 2 \times 3 \times 3 \times 15 \\ &= 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \end{aligned}$$

Therefore $1080 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5$ *which is a product of prime factors*

The greatest Common Factor of two given integers.

We illustrate this concept with an example. To this end consider the numbers 72 and 48. The biggest factor they have in common is called their Greatest Common Factor, (or Greatest Common Divisor). We will write it as $gcf\,(72, 48)$, (g for greatest, c for common, and f for factor). One way of determining it is to identify all their common factors then pick out the largest one. The factors of 72 are

$$2, 3, 4, 5, 6, 8, 9, 12, 18, 24, 36, \text{ and } 72$$

Those of 48 are

$$2, 3, 4, 6, 8, 12, 16, 24 \text{ and } 48$$

Their common factors are $2, 3, 4, 6, 8, 12$ and 24. The largest one is 24, therefore $gcf\,(72, 48) = 24$.

A quicker method runs as follows:

Write 72 as a product of powers of its prime factors:

$$72 = 2^3 \times 3^2$$

Do the same for 48:

$$48 = 2^4 \times 3$$

Identify their common prime factors. They are 2 and 3. The highest power of 2 that is also a common factor is 2^3. The highest power of 3 that is also a common factor is 3. Their product is

$$2^3 \times 3 = 24$$

which is the greatest common factor. Another example:

Example 3 *We determine the greatest common factor of 700 and 1080: First*

$$1080 = 2^3 \times 3^3 \times 5$$

Second

$$700 = 2^2 \times 5^2 \times 7$$

The common prime factors of 1080 *and* 700 *are* 2 *and* 5*. The highest power of 2 that is also a common factor is* 2^2*. The highest power of 5 that is also a common factor is* 5*. Therefore the greatest common factor of* 700 *and 1080 is* $2^2 \times 5 = 20$.

Factoring sums/differences by pulling out common factors

For example, say we want to factor $72 + 48$. One way is to use the greatest common factor for 72 and 48. We found it to be 24, therefore we write

$$\begin{aligned} 72 &= 24 \times 3 \\ 48 &= 24 \times 2 \end{aligned}$$

Now the sum $120 + 450$ may be written as

$$\begin{aligned} 72 + 48 &= 24 \times 3 + 24 \times 2 \\ &= 24\,(3 + 2) = 24 \times 5. \end{aligned}$$

In other words, $72+48 = 24\times 5$, therefore we have factored $72+48$. We will call this the method of factoring by pulling out common factors.
Similarly, $1080 = 20 \times 54$ and $700 = 20 \times 35$ and the sum $1080 + 700$ may be written as

$$\begin{aligned} 1080 + 700 &= 20 \times 54 + 20 \times 35 \\ &= 20\,(54 + 35) = 20 \times 89 \end{aligned}$$

In other words, $1080 + 700 = 20 \times 89$

Exercise 4

1. *Determine the greatest common factor of the given pair of integers*

   (a) 1386 *and* 1820 *(b)* 1620 *and* 1155 *(c)* 1254 *and* 918

2. *The greatest common divisor of 380 and 180 is 20. Use it to factor* $380 - 180$.

3. *The greatest common divisor of 518 and 154 is 14. Use it to factor* $518 + 154$.

4. *The greatest common divisor of 1218 and 510 is 6. Use it to factor* $1218 - 510$.

5. *Show that:*

   (a) $250 + 250 \times 0.05 = 250 \times 1.05$.

   (b) $250 \times 1.05 + (250 \times 1.05) \times 0.05 = 250 \times 1.05^2$.

6. *Use the method of question 5 to factor* $250 \times 1.05^2 + \left(250 \times 1.05^2\right) \times 0.05$.

Greatest Common Factor of two or more algebraic expressions.

We also use examples here to illustrate the concepts. To this end consider the expressions $10x$ and $2x^3$. The factors of $10x$ are

$$2, 5, 10, 2x, 5x, \text{ and } 10x$$

The factors of $2x^3$ are

$$2, 2x, 2x^2 \text{ and } 2x^3$$

Their common factors are 2 and $2x$. Clearly, $2x$ is the "largest" one. Just as we did with whole numbers, we may factor the algebraic expression $10x + 2x^3$ by the method of pulling out common factors. We simply write

$$10x = 2x \times 5 \quad \text{and} \quad 2x^3 = 2x \times x^2.$$

Then the sum $10x + 2x^3$ may be written as

$$\begin{aligned} 10x + 2x^3 &= 2x \times 5 + 2x \times x^2 \\ &= 2x\left(5 + x^2\right) \end{aligned}$$

Therefore, $10x + 2x^3 = 2x\left(5 + x^2\right)$. Use multiplication to check the answer.

Example 5 *To factor $8x + 32$, we first note that the given expression is the sum of the terms $8x$ and 32. The greatest common factor of the two terms is 8; and $8x = 8 \times x$ while $32 = 8 \times 4$. Therefore*

$$\begin{aligned} 8x + 32 &= 8 \times x + 8 \times 4 \\ &= 8\left(x + 4\right). \end{aligned}$$

Example 6 *To factor $12x^4y^2 + 36y^4 - 48y^6$, we note that the given expression is the sum of the terms $12x^4y^2$, $36y^4$, and $-48y^6$. Their greatest common factor is $12y^2$. Now we write*

$$12x^4y^2 = \left(12y^2\right) \times x^4,\ 36y^4 = \left(12y^2\right) \times 3y^2,\ and - 48y^6 = \left(12y^2\right) \times \left(-4y^4\right).$$

Therefore

$$\begin{aligned} 12x^4y^2 + 36y^4 - 48y^6 &= \left(12y^2\right) \times x^4 + \left(12y^2\right) \times 3y^2 + \left(12y^2\right) \times \left(-4y^4\right) \\ &= 12y^2\left(x^4 + 3y^2 - 4y^4\right) \end{aligned}$$

You may use multiplication to check this answer.

Exercise 7 *Factor each of the following by pulling out common factors.*

1. $24x - 18y$
2. $15x^2 + 9x^4$
3. $P + rP$
4. $7uv^2t^3 - 6u^2vt$
5. $12x^2y^3z - 16x^3y^2z^3$
6. $P(1 + r) + P(1 + r)r$
7. $5ab^2 + 2ab - 18a^2b^3$
8. $4x^3y - 12x^2y^2 + 20y^3$
9. $P(1 + r)^2 + P(1 + r)^2r$
10. $P(1 + r)^k + P(1 + r)^kr$