Expansions of Powers of $(A+B)$

We must address these expansions because they are a source of many errors in algebraic manipulations. It is quite common for readers to assume that $(A+B)^2$ is equal to A^2+B^2, that $(A+B)^3$ is equal to A^3+B^3, etc. We state this categorically; **that is not true**. What is true is that

$$\begin{aligned}
(A+B)^2 &= (A+B)(A+B) = A^2+AB+BA+B^2 = A^2+2AB+B^2 \\
(A+B)^3 &= (A+B)^2 = (A+B)(A+B)^2 = (A+B)\left(A^2+2AB+B^2\right) = A^3+3A^2B+3AB^2+B^3 \\
(A+B)^4 &= (A+B)(A+B)^3 = A^4+4A^3B+6A^2B^2+4AB^3+B^4,
\end{aligned}$$

There is a simple pattern one may use to write all these down. The following observations help in determining it:

$$\begin{array}{ccccccccccc}
(A+B)^2 & = & (A+B)(A+B) & = & A^2 & + & AB & & \\
 & & & & & + & AB & + & B^2 \\
 & & & = & A^2 & + & 2AB & + & B^2 \\
 & & & = & A^2 & + & 2AB & + & B^2
\end{array}$$

As a prelude to getting the pattern note that $A^2+2AB+B^2$ is actually a short way of writing $1\cdot A^2+2\cdot AB+1\cdot B^2$ where the dot $\cdot$ represents multiplication. The coefficients 1, 2, 1 in $1\cdot A^2+2\cdot AB+1\cdot B^2$ may be obtained as follows: Write the row 1 1 twice, one below the other, as

$$\begin{array}{cc}
1 & 1 \\
1 & 1
\end{array}$$

Next, shift the second row, to the right, through one position as shown below

$$\begin{array}{ccc}
1 & 1 & \\
 & 1 & 1
\end{array}$$

Now add the numbers in each of the three columns to get 1 2 1 .

Turning to $(A+B)^3$, we use the fact that $(A+B)^2 = A^2+2AB+B^2$ to write it as

$$\begin{array}{ccccccccccc}
(A+B)^3 & = & (A+B)(A+B)^2 & & & & & & & & \\
 & = & (A+B)\left(A^2+2AB+B^2\right) & = & A^3 & + & 2A^2B & + & AB^2 & & \\
 & & & & & + & A^2B & + & 2AB^2 & + & B^3 \\
 & & & = & A^3 & + & 3A^2B & + & 3AB^2 & + & B^3
\end{array}$$

The coefficients 1, 3, 3, 1 in $A^3+3A^2B+3AB^2+B^3$ may be obtained as follows: Write row 1 2 1 twice, one below the other, as

$$\begin{array}{ccc}
1 & 2 & 1 \\
1 & 2 & 1
\end{array}$$

then, (as we did above), shift the lower row to the right through one position, (see below)

$$\begin{array}{cccc}
1 & 2 & 1 & \\
 & 1 & 2 & 1
\end{array}.$$

As expected, add the numbers in the numbers in each of the four columns to get 1 3 3 1 .

Following the above pattern, you should expect the coefficients in the expansion of $(A+B)^4$ to be obtained by writing the row 1 3 3 1 twice, (one below the other), shift the lower row to the right through one position then add the numbers in each of the five columns to get   1   4   6   4   1 . This is, of course true, and

$$(A+B)^4 = A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4$$

The coefficients in the expansion of $(A+B)^5$ are obtained by writing the row   1   4   6   4   1 twice, (one below the other), shift the lower one to the right through one position then add the numbers in each of the six columns to get   1   5   10   10   5   1 . Now the expansion for $(A+B)^5$ is clear. It is

$$(A+B)^5 = A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5$$

Pascal's Triangle

It is now time to introduce Pascal's triangle. It is made of rows of integers, and each row has one more number than the row above it. The first row has two numbers and it is

1   1 .

These are the coefficients of A and B in $A+B$.

The second row has three numbers. The first one is a 1. The second one is the sum of the two numbers in the first row, which is $1+1=2$. The last one is a 1. Therefore the second row is

1   2   1 .

These are the coefficients in the expansion of $(A+B)^2$.

The third row has four numbers. The first one is a 1. The second one is the sum of the *first two numbers* in the second row which is $1+2=3$. The third one is the sum of *the next two numbers* in the second row, which is $2+1=3$. The fourth one is a 1. Therefore the third row is

1   3   3   1 .

These are the coefficients in the expansion of $(A+B)^3$

The fourth row has five numbers. The first one is a 1. The second one is the sum of the *first two numbers* in the third row, which is $1+3=4$. The third one is the sum of the *next two numbers* in the third row, which is $3+3=6$. The fourth one is the sum of the *next two numbers* in the third row, which is $3+1=4$. The fifth one is a 1. Therefore the fourth row is

1   4   6   4   1 .

These are the coefficients in the expansion of $(A+B)^4$.

The fifth row has six numbers and they are the coefficients in the expansion of $(A+B)^5$. It should be clear how to get them. The first one is a 1. The second one is 5, (the sum of 1 and 4). The third one is 10, (the sum of 4 and 6). The fourth one is also 10, (the sum of 6 and 4). The fifth one is 5, (the sum is 4 and 1), and the last one is a 1. Thus the fifth row is

1   5   10   10   5   1 .

Show that the sixth row is

1   6   15   20   15   6   1

These are the coefficient in the expansion of $(A+B)^6$. Therefore

$$(A+B)^6 = A^6 + 6A^5B + 15A^4B^2 + 20A^3B^3 + 15A^2B^4 + 6AB^5 + B^6$$

The first 6 rows of Pascal's triangle are

$$\begin{array}{ccccccccccccc} & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

Problem 1 *Add two more rows to the above Pascal's triangle and use one of them to expand* $(A+B)^7$.

Example 2 *To expand* $\left(\frac{x}{2}-\frac{4}{3}\right)^2$.

Solution: We take $A=\frac{x}{2}$, $B=-\frac{4}{3}$ *and substitute into* $(A+B)^2=A^2+2AB+B^2$. *The result is*

$$\left(\frac{x}{2}-\frac{4}{3}\right)^2=\left(\frac{x}{2}\right)^2+2\left(\frac{x}{2}\right)\left(-\frac{4}{3}\right)+\left(-\frac{4}{3}\right)^2=\frac{x^2}{4}-\frac{4x}{3}+\frac{16}{9}$$

Example 3 *To expand* $\left(\frac{a}{b}-\frac{2b}{a}\right)^3$.

Solution: We take $A=\frac{a}{b}$, $B=\frac{2b}{a}$ *and substitute into* $(A+B)^3=A^3+3A^3B+3AB^2+B^3$. *The result is*

$$\begin{aligned}\left(\frac{a}{b}-\frac{2b}{a}\right)^3 &= \left(\frac{a}{b}\right)^3+3\left(\frac{a}{b}\right)^2\left(-\frac{2b}{a}\right)+3\left(\frac{a}{b}\right)\left(-\frac{2b}{a}\right)^2+\left(-\frac{2b}{a}\right)^3\\ &= \frac{a^3}{b^3}-\frac{6a}{b}+\frac{12b}{a}-\frac{8b^3}{a^3}\end{aligned}$$

Example 4 *To expand* $(5+3x-y)^2$.

Solution: We may write this as $[5+(3x-y)]^2$ *then take* $A=5$ *and* $B=(3x-y)$ *and substitute into* $(A+B)^2=A^2+2AB+B^2$. *The result is*

$$\begin{aligned}[5+(3x-y)]^2 &= 5^2+2(5)(3x-y)+(3x-y)^2\\ &= 25+10(3x-y)+(3x)^2+3(3x)(-y)+(-y)^2\\ & \quad 25+30x-10y+9x^2-9xy+y^2\end{aligned}$$

Example 5 *What is the coefficient of* x *in the expansion of* $\left(\frac{2}{x}+x\right)^5$?

Solution: We need to use the 5th row of Pascal's triangle to get the terms in the expansion of $\left(\frac{2}{x}+x\right)^5$. *They are* $\left(\frac{2}{x}\right)^5$, $5\left(\frac{2}{x}\right)^4x$, $10\left(\frac{2}{x}\right)^3x^2$, $10\left(\frac{2}{x}\right)^2x^3$, $5\left(\frac{2}{x}\right)x^4$ *and* x^5. *The* x *term among them is* $10\left(\frac{2}{x}\right)^2x^3$ *which may b reduced to* $40x$. *Therefore the coefficient of* x *in the expansion is* 40.

Exercise 6

1. *Simplify the given expression and give an answer with no negative exponent(s):*

   (a) $\left(-3x^5y^{-5}\right)\left(2x^7y^2\right)$ (b) $\dfrac{16a^8b^3c^{-2}}{6a^2b^{-2}c^4}$ (c) $\dfrac{75ab^5c^9}{25a^4b^{-3}c^3}$ (d) $\left(\dfrac{x^2y^3z^5}{x^{-6}y^{-5}z^{-3}}\right)^{-2}$

2. *Expand the given expression and simplify as much as possible.*

   (a) $(x+2y)^2$ (b) $\left(\dfrac{a}{2b}-\dfrac{2a}{3b}\right)^3$ (c) $[3+(2a-3b)]^2$ (d) $\left(a^2-2b^3\right)^2$

3. *Show that*

   $$a^2+ab+b^2=\tfrac{1}{2}(a+b)^2+\tfrac{1}{2}a^2+\tfrac{1}{2}b^2$$

   Give a similar expression for a^2-ab+b^2

4. *Show that* $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$

5. *Expand* $(a-b+c)^2$