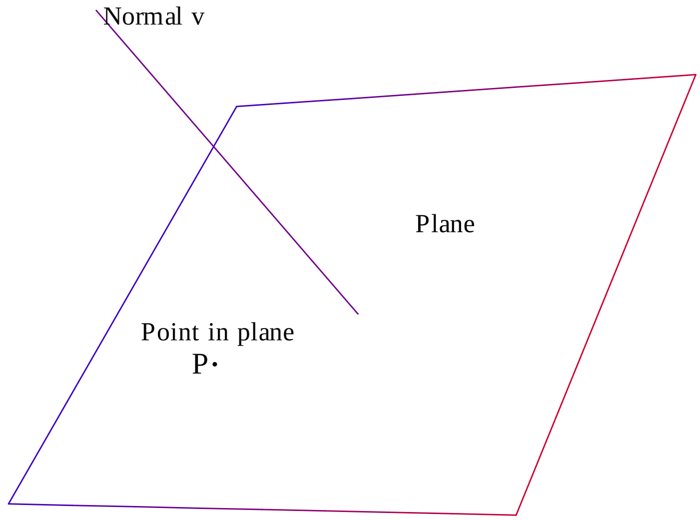
Planes in Space

A plane in space is uniquely specified by giving a point P in the plane and a vector $\mathbf{v}$, (called a normal to the plane), that is orthogonal to every vector in the plane.



Say the plane passes through a point $P\,(a,b,c)$ and $\mathbf{v} =< v_1, v_2, v_3 >$ is a vector that is orthogonal to every vector in the plane. Let $Q(x,y,z)$ be an arbitrary point in the plane. An equation connecting x, y and z is, by definition, the equation of the plane. To get it, note that $\overrightarrow{PQ} =< x-a, y-b, c-z >$ is a vector in the plane, therefore it must be orthogonal to $\mathbf{v}$. We know that the dot product of two orthogonal vectors is zero, therefore

$$\overrightarrow{PQ} \cdot \mathbf{v} = \mathbf{0}$$

In other words, $(x-a)\,v_1 + (y-b)\,v_2 + (c-z)\,v_3 = 0$. We may expand and write the result as

$$v_1 x + v_2 y + v_3 z = \lambda$$

where $\lambda = v_1 a + v_2 b + v_3 c$ is a constant. This is the standard equation of the plane. Note that the coefficients v_1 of x, v_2 of y and v_3 of z are the components of the vector $\mathbf{v} =< v_1, v_2, v_3 >$ that is orthogonal to every vector in the plane.

Example 1 *To find the equation of the plane that contains the points* $P\,(2,-3,5)$*,* $Q\,(0,-6,7)$ *and* $R(1,-2,4)$*, we may use any one of the 3 given points as a point in the plane. We also need a vector that is perpendicular to every vector* $\mathbf{v}$ *in the plane. To get it, consider the vectors* $\mathbf{u} = \overrightarrow{PQ} =< -2,-3,2 >$ *and* $\mathbf{w} = \overrightarrow{PR} =< -1,1,-1 >$*. They are in the plane, therefore their cross product is orthogonal to every vector in the plane. By definition,*

$$\mathbf{u}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -3 & 2 \\ -1 & 1 & -1 \end{vmatrix} = \mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$$

Therefore we must determine the equation of the plane that contains the point $P\,(2,-3,5)$ *and is orthogonal to* $\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$*. As described above, we argue that if* $S(x,y,z)$ *is a point in the plane then* $\overrightarrow{PS}$ *is orthogonal to the vector* $\mathbf{i} - 4\mathbf{j} - 5\mathbf{k} = < 1,-4,-5 >$*, hence*

$$(x-2)\,(1) + (y+3)\,(-4) + (z-5)\,(-5) = 0$$

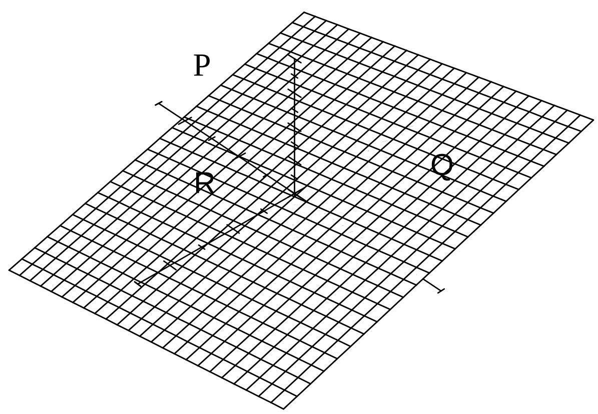
I.e. $x - 4y - 5z = 2 + 12 - 25$. *This may be simplified to* $x - 4y - 5z = -11$. *[You can easily check that the given points* $(2, -3, 5)$, $(0, -6, 7)$ *and* $(1, -2, 4)$ *satisfy this equation.]*

Exercise 2

1. *Let* $A(a_1, b_1, c_1)$ *and* $B(a_2, b_2, c_2)$ *be distinct points in space. Let* ρ *be the plane with the property that if* P *is any point in* ρ *then the line segments* PA *and* PB *have the same length. Determine the equation of* ρ.

2. *Show that the line with symmetric equations* $\dfrac{x-4}{2} = \dfrac{y-5}{3} = \dfrac{z-5}{2}$ *intersects the line with symmetric equations* $\dfrac{x-9}{3} = \dfrac{y+4}{-1} = \dfrac{z-7}{2}$ *and determine the equation of the plane that contains them. (Hint: Use vectors parallel to the given lines to determine a normal to the required plane.)*

3. *Let* $ax + by + cz = \lambda$ *be a given plane and* $P(e_1, e_2, e_3)$ *a given point in space. In this exercise, you calculate the distance from* P *to the plane. To this end, take any point* $Q(x, y, z)$ *in the plane. The required distance is* $|PR|$ *where* R *is the point in the plane such that angle* PRQ *is a right angle.*



   Let $\mathbf{n}$ *be a normal to the plane. Then* $\vec{PR}$ *is parallel to* $\mathbf{n}$.

   (a) *Use this information and the result of exercise* **??** *on page* **??** *to show that the required distance is*

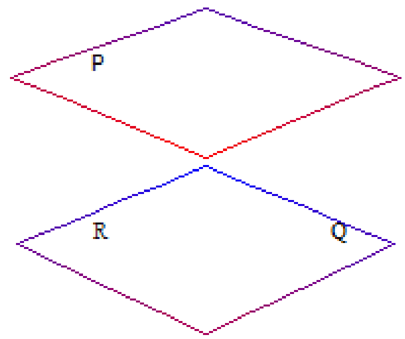
$$\frac{\left|\vec{PQ} \cdot \mathbf{n}\right|}{||\mathbf{n}||} \tag{1}$$

   (b) *Verify that* $\dfrac{\left|\vec{PQ} \cdot \mathbf{n}\right|}{||\mathbf{n}||} = \dfrac{|ae_1 + be_2 + ce_3 - \lambda|}{\sqrt{a^2 + b^2 + c^2}}$ *then deduce that the distance from* P *to the plane is zero if and only if* P *is in the plane.*

4. *You are required to determine an expression for the distance between two parallel planes. By definition, the angle between two planes is the angle between their respective normals. Explain why the equations of two parallel planes* ρ_1 *and* ρ_2 *may be assumed to have the form*

$$\rho_1 : \quad ax + by + cz = \alpha \qquad \textit{and} \qquad \rho_2 : \quad ax + by + cz = \beta$$

where a, b, c, α *and* β *are constants. Let* $P(x_1, y_1, z_1)$ *be a point in the plane* ρ_1 *and* $Q(x_2, y_2, z_2)$ *be a point in plane* ρ_2*. Drop a perpendicular* PR *from* P *to a point* R *in the plane* ρ_2*. Let* $\mathbf{n} = < a, b, c >$.



(a) *Show that the required distance, (remember that distances cannot be negative), is*

$$\left| \frac{\overrightarrow{PQ} \cdot \mathbf{n}}{||\mathbf{n}||} \right|$$

(b) *Evaluate* $\overrightarrow{PQ} \cdot \mathbf{n}$ *and use the fact that* $ax_1 + by_1 + cz_1 = \alpha$ *and* $ax_2 + by_2 + cz_2 = \beta$ *to derive a simpler expression for the required distance.*

5. *The angle between two planes is, by definition, the angle between their respective normals. Show that the planes* $2x + y - z = 1$ *and* $x - y + 3z = 5$ *are not parallel then give a parametric equation of the line in which they intersect.*

6. *Consider the line with symmetric equations* $\dfrac{x-1}{2} = \dfrac{y+3}{-2} = \dfrac{z+2}{3}$*. There are infinitely many vectors perpendicular to it. To determine all of them, simply note that they all lie in a plane perpendicular to the line. Thus determine the equation of the plane through* $(1, -3, -2)$ *perpendicular to the line. Now give an example of two vectors perpendicular to the line.*

7. *Consider the plane containing the point* $P(3, 3, 5)$ *and the line with symmetric equation* $\dfrac{x+2}{3} = \dfrac{y-4}{5} = \dfrac{z-2}{2}$.

   (a) *Determine a vector joining* P *and a point on the line.*

   (b) *Determine a vector parallel to the line*

   (c) *Determine the equation of the plane containing* $P(3, 3, 5)$ *and the given line.*