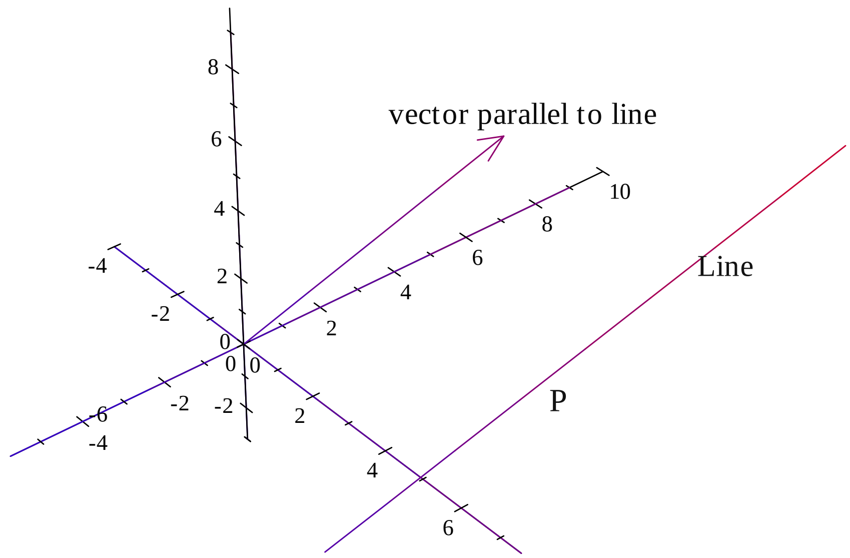
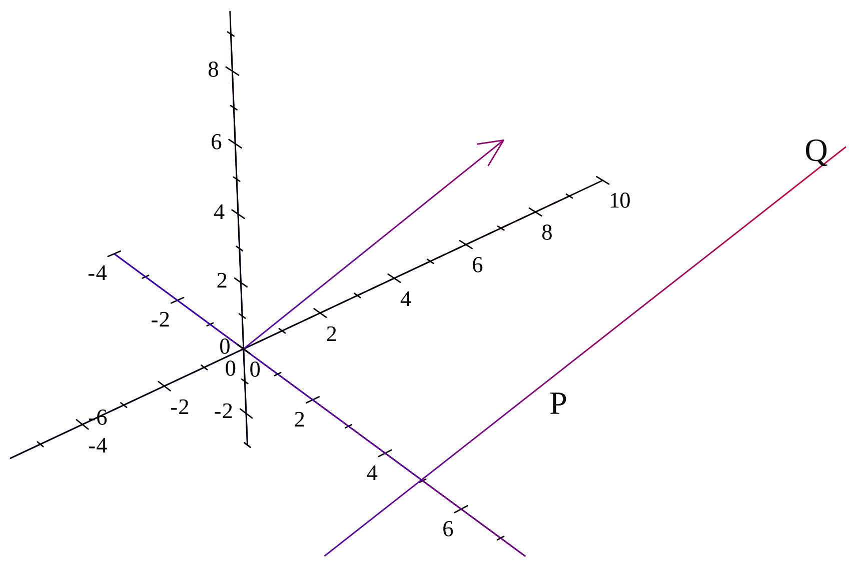
Lines in Space

To pin down a straight line in space, it suffices to give one point $P\,(a, b, c)$ on the line *and* specify a direction one would face if one were to walk along line, by giving a vector pointing in that direction.



Say a line L passes through a point $P\,(a, b, c)$ and it is parallel to a vector $\mathbf{u} = < u_1, u_2, u_3 >$. Let $Q(x, y, z)$ be any point on L.



The equations connecting x, y and z are, by definition, the equations of L. To determine them, simply note that $\overrightarrow{PQ}$ is parallel to $\mathbf{u}$, therefore it is a scalar multiple of $\mathbf{u}$. Of course $\overrightarrow{PQ} = < x - a, y - b, x - c >$, therefore there is a real number t such that

$$< x - a, y - b, z - c > = t < u_1, u_2, u_3 > = < tu_1, tu_2, tu_3 >$$

In component form this translates into $x-a=tu_1$, $y-b=tu_2$ and $z-c=tu_3$. We may rearrange these 3 as

$$\begin{aligned} x &= a+tu_1 \\ y &= b+tu_2 \qquad \text{or simply} \qquad (x,y,z)=(a+tu_1,b+tu_2,c+tu_3) \\ z &= c+tu_3 \end{aligned}$$

These are called the parametric equations of L. The parameter is t and as it varies among the set of real numbers, the three equations give the different points (x,y,z) on L. For example, $t=0$ gives the point $P(a,b,c)$.

If u_1, u_2, and u_3 are all nonzero then we may eliminate t to get

$$\frac{x-a}{u_1}=\frac{y-b}{u_2}=\frac{z-c}{u_3}$$

These are called the symmetric equations of L. Note that the denominators are the components of a vector parallel to the line.

Example 1 *The line that passes through* $(4,-3,-5)$ *and is parallel to the vector* $<-2,-1,6>$ *has parametric equations*

$$\begin{aligned} x &= 4-2t \\ y &= -3-t \\ z &= -5+6t \end{aligned}$$

Its symmetric symmetric equations are

$$\frac{x-4}{-2}=\frac{y+3}{-1}=\frac{z+5}{6}.$$

Example 2 *Say we want to find the equations of the line L that passes through the two points $P(1,1,-3)$ and $Q(3,5,6)$. Any one of $(1,1,-3)$ and $(3,5,6)$ may serve as a point on L. Say we choose to use P. For a vector parallel to L, we may take $\overrightarrow{PQ}=<2,4,9>$. Therefore the symmetric equations of L are*

$$\frac{x-1}{2}=\frac{y-1}{4}=\frac{z+3}{9}.$$

Its parametric equations are

$$\begin{aligned} x &= 1+2t \\ y &= 1+4t \\ z &= -3+9t \end{aligned}$$

Example 3 *To show that the line L_1 with symmetric equations $\frac{x-3}{2}=\frac{y+5}{-3}=\frac{z-4}{1}$ intersects the line L_2 with symmetric equations $\frac{x+2}{3}=\frac{y+5}{3}=\frac{z-5}{-2}$.*

The parametric form of L_1 is $(2t+3,-3t-5,t+4)$, $t\in\mathbb{R}$ and that of L_2 is $(3s-2,3s-5,-2s+5)$, $s\in\mathbb{R}$. Therefore we have to show that there is a value of t and a value of s that give the same point in space. Such a pair must satisfy the conditions

$$\begin{aligned} 2t+3 &= 3s-2 \\ -3t-5 &= 3s-5 \\ t+4 &= -2s+5 \end{aligned}$$

These are 3 equations with two unknowns, therefore if they have a solution, it may be obtained by solving any two of them. We solve $2t+3=3s-2$ and $t+4=-2s+5$. The solution is $s=1$ and $t=-1$. (Verify that any other two of the three equations give the same solution.) Therefore the two lines intersect at $(-2\times 1+3,-3\times(-1)-5,-1+4)=(1,-2,3)$

Exercise 4

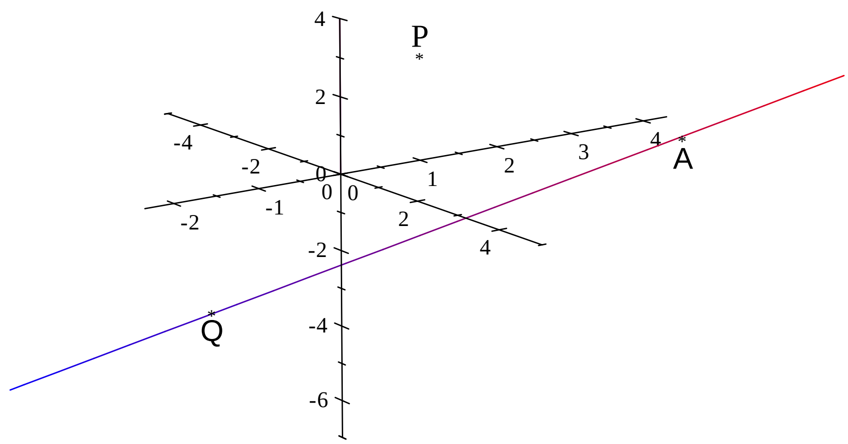
1. *A line perpendicular to two given non-parallel vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *must be parallel to the cross product* $\mathbf{u}\times\mathbf{v}$*. Use this observation to find the equation of the line through* $(-2,-1,5)$ *which is also perpendicular to the vectors* $< 1, 0, 2 >$ *and* $< 3, 4, -2 >$*.*

2. *The angle between a line* ℓ_1 *parallel to a vector* $\mathbf{n}$ *and a line* ℓ_2 *parallel to a vector* $\mathbf{m}$ *is, by definition, the angle between* $\mathbf{n}$ *and* $\mathbf{m}$*.*

   (a) *Show that the line with symmetric equations* $\dfrac{x-4}{2} = \dfrac{y-5}{3} = \dfrac{z-5}{2}$ *intersects the line with symmetric equations* $\dfrac{x-9}{3} = \dfrac{y+4}{-1} = \dfrac{z-7}{2}$ *and determine the angle between the two lines.*

   (b) *Give an example of two lines in space, (called skew lines), that are not parallel AND do not intersect. You need to verify that they are not parallel and that they do not intersect.*

3. *Fill in the missing details in the following steps to determine the distance from a given point* $P\,(e_1, e_2, e_3)$ *to a given line with symmetric equations* $\dfrac{x-a}{u_1} = \dfrac{y-b}{u_2} = \dfrac{z-c}{u_3}$*.*



The line passes through the point $A(a, b, c)$ *and is parallel to the vector* $\mathbf{u} =< u_1, u_2, u_3 >$*. A point* Q *on the given line has coordinates* $(tu_1 + a, tu_2 + b, tu_3 + c)$*,* $t \in \mathbb{R}$*. Let* $|PQ|$ *be the length of the line segment from* P *to* Q*. The problem is to find the minimum value of* $|PQ|$*. Show that*

$$|PQ|^2 = (tu_1 + a - e_1)^2 + (tu_2 + b - e_2)^2 + (tu_3 + c - e_3)^2 . \qquad (1)$$

Let $\mathbf{v} = \overrightarrow{AP}$*. By expanding the terms* $(tu_1 + a - e_1)^2$*,* $(tu_2 + b - e_2)^2$ *and* $(tu_3 + c - e_3)^2$*, show that (1) may be written as* $|PQ|^2 = ||\mathbf{u}||^2\, t^2 - 2\mathbf{u}\cdot\mathbf{v}t + ||\mathbf{v}||^2$ *.*

Determine the critical point of the function

$$g(t) = ||\mathbf{u}||^2\, t^2 - 2\mathbf{u}\cdot\mathbf{v}t + ||\mathbf{v}||^2$$

and use the second derivative test to verify that it is a point of relative minimum. Deduce that the shortest distance from P *to the line is*

$$d = \frac{\sqrt{||\mathbf{u}||^2\, ||\mathbf{v}||^2 - (\mathbf{u}\cdot\mathbf{v})^2}}{||\mathbf{u}||} \qquad (2)$$

4. *Use formula (2) to calculate the distance:*

(a) *From the point* $(1,0,8)$ *to the line* $\dfrac{x-3}{2} = \dfrac{y-4}{3} = \dfrac{z+1}{-1}$.

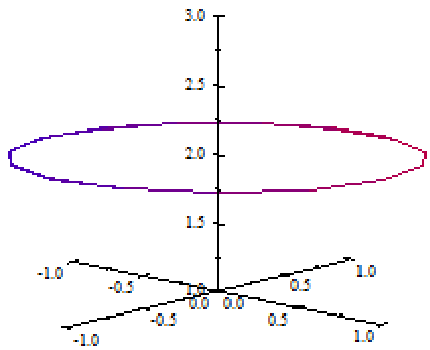
(b) *From the point* $(-2,1,-3)$ *to the line that passes through* $(0,2,0)$ *and* $(1,4,5)$.

5. *Show that if the point* P *in question 3 is on the given line then* d *in formula (2) is zero.*

More General Curves in Space

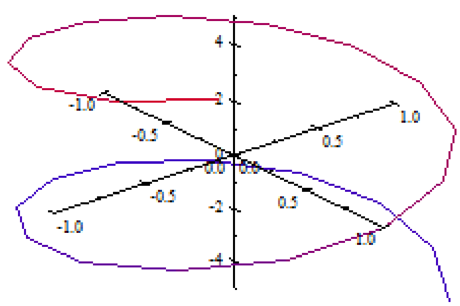
A general curve in space is specified parametrically by giving the coordinates of its points as functions of one variable, say t.

Example 5 *Consider the set of points* $(x, y, z) = (\cos t, \sin t, 2)$, $0 \le t \le 2\pi$. *Thus,* $x = \cos t$, $y = \sin t$ *and* $z = 2$, *hence* $x^2 + y^2 = 2$ *and* z *has the constant value 2. They lie on a circle with center* $(0, 0, 2)$ *and radius* $\sqrt{2}$.



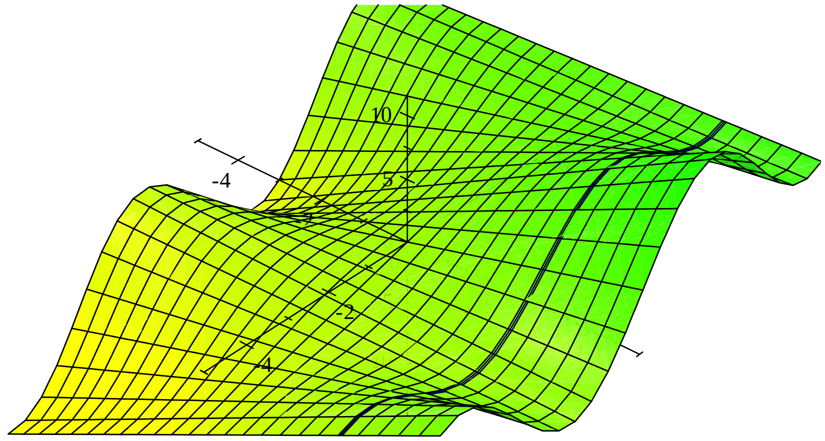
We may also describe this curve as the set of points in space with position vectors $\mathbf{c}(t) = <\cos t, \sin t, 2>$, $t \in [0, 2\pi]$.

Example 6 *The curve consisting of the set of points in space with position vectors* $\mathbf{c}(t) = <\cos t, \sin t, t>$, $t \in \mathbb{R}$ *is called a helix. A section of its graph is given below.*



Example 7 *Let* $f(x, y)$ *be a given function of two variables. If* (a, b) *is a point in the domain of* f *then the*

"$x=a$ section of f", (consisting of all the possible points $(a, y, f(a, y))$), is a curve in space.



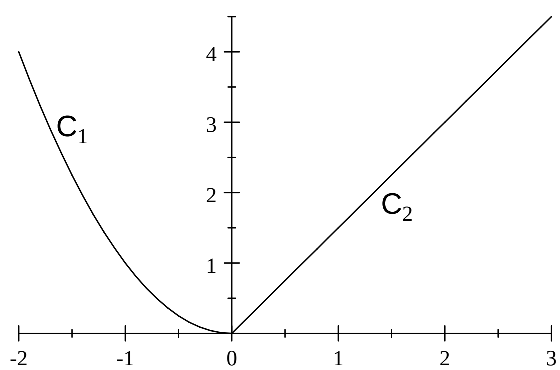
An x-section of $f(x, y) = x + y + x \sin y$

So is the $y = b$ section of f. It consists of all the possible points $(x, b, f(x, b))$

Some terminology

Let $c(t) =< u(t), v(t), w(t) >$, $a \le t \le b$ be a curve. We say that it is a curve from the point $c(a)$ to the point $c(b)$. Denote it by C.

1. The same set of points but traversed from $c(b)$ to $c(a)$ is denoted by $-C$.
2. If $c(a) = c(b)$ then we say that C is a closed curve. Thus the circle in Example 5 is closed.
3. If $u(t)$, $v(t)$, $w(t)$ have continuous derivatives $u'(t)$, $v'(t)$, $w'(t)$ then C is called a smooth curve.
4. A curve may consist of several segments which are parametrized differently. An example is the curve C consisting of the section of the parabola $y = x^2$ on the interval $[-2, 0]$ and the line segment from $(0, 0)$ to $(3, 8)$. Denote the parabola by C_1 and the line segment by C_2.



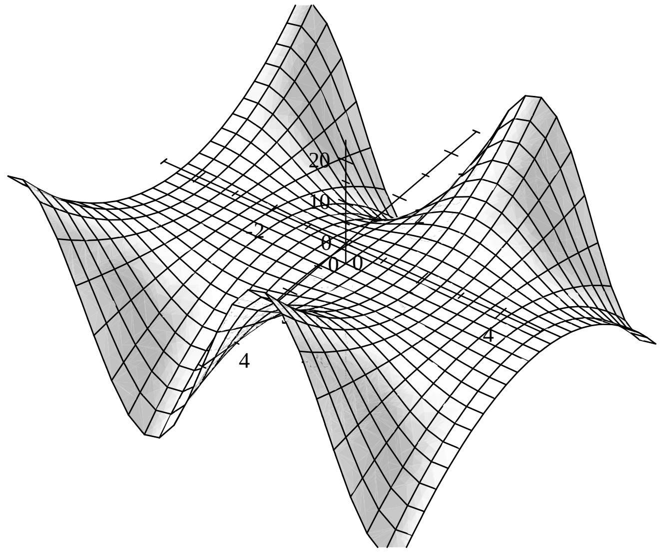
Then we say that C is the sum of C_1 and C_2 which we write as

$$C = C_1 + C_2$$

If C_1 and C_2 are smooth then $C = C_1 + C_2$ is called a piecewise-smooth curve.

Exercise 8

1. Part of the graph of $f(x,y) = x - 5 + x^2 \sin y$ is given below.



Draw, on the graph:

(a) The $x = -2$ section of f.

(b) The $y = -\frac{1}{2}\pi$ section of f.

(c) The $y = \frac{1}{2}\pi$ section of f.