The Dot Product of Two Vectors

Given a pair of vectors $\vec{OP} = \mathbf{u} = < u_1, u_2, u_3 >$ and $\vec{OQ} = \mathbf{v} = < v_1, v_2, v_3 >$, we often want to calculate the angle between them. The figure below shows such a pair in space.

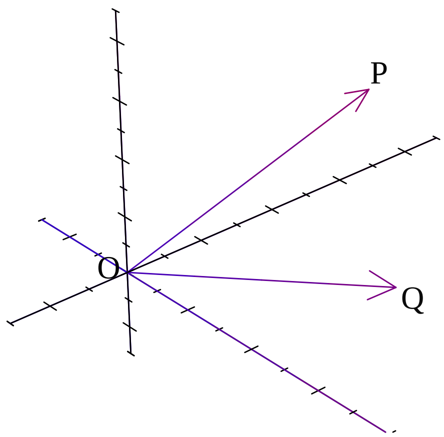


Figure (i)

A specific pair $\mathbf{u} = \vec{OP}$ and $\mathbf{v} = \vec{OQ}$ in a plane is shown below. The coordinates of P and Q are $(-1, 5)$ and $(3, 2)$ respectively.

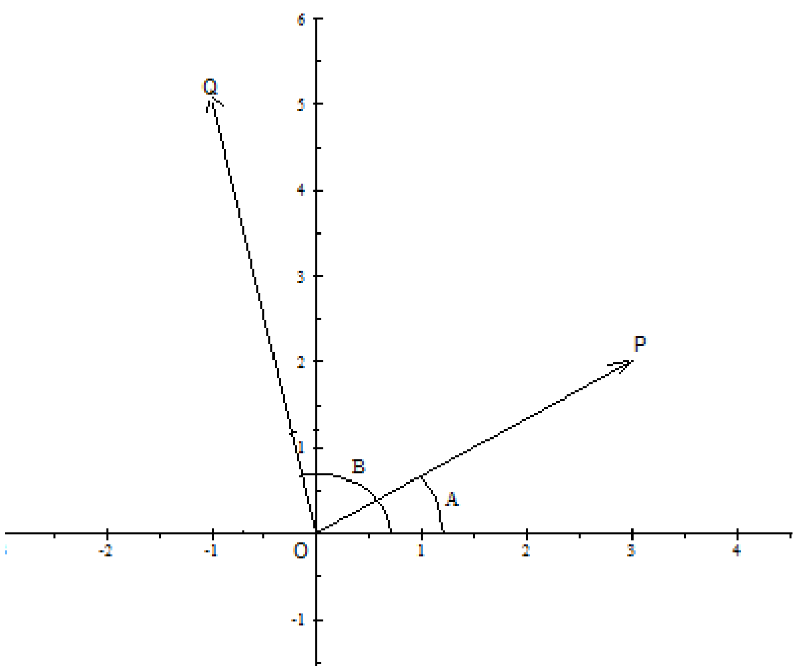


Figure (ii)

The angle between the two vectors is $B - A$. We may calculate it using the identity

$$\cos(B - A) = \cos A \cos B + \sin A \sin B$$

The geometry of the figure gives

$$\sin A = \frac{3}{||\mathbf{u}||},\ \cos A = \frac{2}{||\mathbf{u}||},\ \sin B = \frac{5}{||\mathbf{v}||} \text{ and } \cos B = \frac{-1}{||\mathbf{v}||}$$

Therefore

$$\cos(B-A) = \left(\frac{2}{||\mathbf{u}||}\right)\left(\frac{-1}{||\mathbf{v}||}\right) + \left(\frac{3}{||\mathbf{u}||}\right)\left(\frac{5}{||\mathbf{v}||}\right) = \frac{(2)(-1)+(3)(5)}{||\mathbf{u}||\,||\mathbf{v}||} = \frac{13}{\sqrt{26\times 13}}$$

Thus the angle between them is $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$.

In general, if the coordinates of P and Q are (u_1, u_2) and (v_1, v_2) respectively then

$$\sin A = \frac{u_2}{||\mathbf{u}||},\ \ \cos A = \frac{u_1}{||\mathbf{u}||},\ \ \sin B = \frac{v_2}{||\mathbf{v}||}\ \text{and}\ \cos B = \frac{v_1}{||\mathbf{v}||}$$

Therefore the cosine of the angle $B-A$ between them is given by

$$\cos(B-A) = \left(\frac{u_1}{||\mathbf{u}||}\right)\left(\frac{v_1}{||\mathbf{v}||}\right) + \left(\frac{u_2}{||\mathbf{u}||}\right)\left(\frac{v_2}{||\mathbf{v}||}\right) = \frac{u_1v_1+u_2v_2}{||\mathbf{u}||\ ||\mathbf{v}||}$$

To handle the general case of an angle between two vectors in space, refer back to figure (i) above. Let θ be the angle between them and $\mathbf{w} = \overrightarrow{PR}$ be the vector with initial point (u_1, u_2, u_3) and terminal point (v_1, v_2, v_3). The triangle OPR has sides with lengths $||\mathbf{u}||$, $||\mathbf{v}||$ and $||\mathbf{w}||$. By the law of cosines,

$$||\mathbf{w}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\,||\mathbf{u}||\ ||\mathbf{v}||\cos\theta$$

Therefore

$$\cos\theta = \frac{||\mathbf{u}||^2 + ||\mathbf{v}||^2 - ||\mathbf{w}||^2}{2\,||\mathbf{u}||\ ||\mathbf{v}||}.$$

Note that $||\mathbf{u}||^2 = u_1^2+u_2^2+u_3^2$, $||\mathbf{v}||^2 = v_1^2+v_2^2+v_3^2$ and $||\mathbf{w}||^2 = (u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2$. Expand $(u_1-v_1)^2$, $(u_2-v_2)^2$ and $(u_3-v_3)^2$ then substitute into the numerator of the fraction

$$\frac{||\mathbf{u}||^2 + ||\mathbf{v}||^2 - ||\mathbf{w}||^2}{2\,||\mathbf{u}||\ ||\mathbf{v}||}.$$

The result should be

$$\cos\theta = \frac{u_1v_1+u_2v_2+u_3v_3}{||\mathbf{u}||\ ||\mathbf{v}||}$$

This is a very easy expression to remember. The numerator is obtained by multiplying the corresponding components of $\mathbf{u}$ and $\mathbf{v}$, (the first by the first, the second by the second, the third by the third), then add the results. The denominator is the product of the norms of $\mathbf{u}$ and $\mathbf{v}$. The numerator $u_1v_1+u_2v_2+u_3v_3$ is given a special name. It is called the dot product of $\mathbf{u}$ and $\mathbf{v}$ and it is denoted by $\mathbf{u}\cdot\mathbf{v}$. Thus, by definition

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1+u_2v_2+u_3v_3$$

It is also common to refer to $\mathbf{u}\cdot\mathbf{v} = u_1v_1+u_2v_2+u_3v_3$ as the scalar product of u and v, (since $u_1v_1+u_2v_2+u_3v_3$ is a scalar, not a vector).

Example 1 *The angle θ between $\mathbf{u} = <4,3,-1>$ and $\mathbf{v} = <-1,2,6>$ is given by*

$$\cos\theta = \frac{(4)(-1)+(3)(2)+(-1)(6)}{\left(\sqrt{16+9+1}\right)\left(\sqrt{1+4+36}\right)} = \frac{-4}{\left(\sqrt{26}\right)\left(\sqrt{41}\right)}.$$

Therefore $\theta = \cos^{-1}\left(\frac{-4}{\left(\sqrt{26}\right)\left(\sqrt{41}\right)}\right) = 97°$ *(approximately).*

The properties of a dot product we will use repeatedly are given by the following theorem:

Theorem 2 *Let $\mathbf{u} = <u_1,u_2,u_3>$, $\mathbf{v} = <v_1,v_2,v_3>$ and $\mathbf{w} = <w_1,w_2,w_3>$ be vectors and λ be a real number. Then*

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$, *(i.e. the dot product operation is commutative).*

2. $\mathbf{u} \cdot (\lambda \mathbf{v}) = \lambda \mathbf{u} \cdot \mathbf{v}$

3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, *(i.e. the dot product operation is distributive with respect to addition).*

4. $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$

5. *If the angle θ between $\mathbf{u}$ and $\mathbf{v}$ is a right angle then $\mathbf{u} \cdot \mathbf{v} = 0$. Conversely, if $\mathbf{u}$ and $\mathbf{v}$ are nonzero and $\mathbf{u} \cdot \mathbf{v} = 0$ then θ is a right angle.*

Proof. To verify these properties, we simply use the definition of a dot product.

(a) Since multiplication of real numbers is commutative

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \mathbf{v} \cdot \mathbf{u}$$

This verifies property 1.

(b) By definition, $\lambda \mathbf{v} =< \lambda v_1, \lambda v_2, \lambda v_3 >$, therefore

$$\mathbf{u} \cdot (\lambda \mathbf{v}) = u_1 \lambda v_1 + u_2 \lambda v_2 + u_3 \lambda v_3 = \lambda < u_1 v_1 + u_2 v_2 + u_3 v_3 >= \lambda \mathbf{u} \cdot \mathbf{v}$$

which verifies property 2.

(c) By definition, $\mathbf{v} + \mathbf{w} =< v_1 + w_1, v_2 + w_2, v_3 + w_3 >$. It follows that

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3) \\ &= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 + u_3 v_3 + u_3 w_3 \\ &= (u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_1 w_1 + u_2 w_2 + u_3 w_3) \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \end{aligned}$$

(d) $\mathbf{u} \cdot \mathbf{u} = u_1 u_1 + u_2 u_2 + u_3 u_3 = u_1^2 + u_2^2 + u_3^2 = ||\mathbf{u}||^2$ thus verifying property 4.

(e) If the angle θ between $\mathbf{u}$ and $\mathbf{v}$ is a right angle then $\cos\theta = 0$. It follows from the identity

$$\cos\theta = \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{||\mathbf{u}||\ ||\mathbf{v}||},$$

that $u_1 v_1 + u_2 v_2 + u_3 v_3 = 0$. In other words, $\mathbf{u} \cdot \mathbf{v} = 0$. Conversely, if $u_1 v_1 + u_2 v_2 + u_3 v_3 = 0$ then $\cos\theta = 0$, which implies that θ is a right angle.

■

Property 5 gives an easy method of determining whether or not two given vectors are orthogonal, (i.e. the angle between them is a right angle). Just check whether or not their dot product is 0.

Exercise 3

1. *Compute the acute angle between the two given vectors:*

   (a) $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ *and* $\mathbf{v} = \mathbf{i} + 2\mathbf{k}$.

   (b) $\mathbf{u} = -2\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$ *and* $\mathbf{v} = \mathbf{i} - \mathbf{j} - 0.5\mathbf{k}$

   (c) $\mathbf{u} = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$ *and* $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

2. *There are infinitely many vectors that are perpendicular to* $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k.m}$. *Give one of them.*

3. *Show that the points $A(1, -1, 2)$, $B(5, 1, 1)$ and $C(4, -5, 6)$ are vertices of a right triangle, (i.e. a triangle with a right angle). Is it an isosceles triangle?*

4. *Find two vectors $< a_1, a_2, a_3 >$ and $< b_1, b_2, b_3 >$, (they are both arrows originating from the origin), which have the same norm and are orthogonal. Translate them by the same vector and use the result to give an example of an isosceles right triangle none of whose vertex is at the origin.*

5. *Use the fact that* $|\mathbf{u}\cdot\mathbf{v}| = |\,||\mathbf{u}||\;||\mathbf{v}||\cos\theta|$ *to deduce that* $|\mathbf{u}\cdot\mathbf{v}| \le ||\mathbf{u}||\;||\mathbf{v}||$ *for all vectors* $\mathbf{u}$ *and* $\mathbf{v}$*. Under what conditions is* $|\mathbf{u}\cdot\mathbf{v}| = ||\mathbf{u}||\;||\mathbf{v}||$*?*

6. *Justify the following steps to prove the triangle inequality which states that* $||\mathbf{u}+\mathbf{v}|| \le ||\mathbf{u}|| + ||\mathbf{v}||$ *for all vectors* $\mathbf{u}$ *and* $\mathbf{v}$*:*

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \qquad (\textit{why?}) \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{2u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ &= ||\mathbf{u}||^2 + 2\mathbf{u}\cdot\mathbf{v} + ||\mathbf{v}||^2 \qquad (\textit{why?}) \\ &\le ||\mathbf{u}||^2 + 2\,||\mathbf{u}||\;||\mathbf{v}|| + ||\mathbf{v}||^2 \qquad (\textit{why?}) \\ &= (||\mathbf{u}|| + ||\mathbf{v}||)^2 \end{aligned}$$

Now deduce that $||\mathbf{u}+\mathbf{v}|| \le ||\mathbf{u}|| + ||\mathbf{v}||$.

(a) *Under what conditions is* $||\mathbf{u}+\mathbf{v}|| = ||\mathbf{u}|| + ||\mathbf{v}||$*?*

(b) *Given two vectors* $\mathbf{u}$ *and* $\mathbf{v}$*, we may write* $||\mathbf{u}|| = ||(\mathbf{u}-\mathbf{v})+\mathbf{v}||$*. Use this to deduce that*

$$||\mathbf{u}|| - ||\mathbf{v}|| \le ||\mathbf{u}-\mathbf{v}||\,.$$

(Hint: Apply the triangle inequality to $||(\mathbf{u}-\mathbf{v})+\mathbf{v}||$ *then rearrange.)*

7. *Given a vector* $\mathbf{v}$ *and a unit vector* $\mathbf{u}$*, the scalar* $\mathbf{v}\cdot\mathbf{u}$ *is called the component of* $\mathbf{v}$ *in the direction of* $\mathbf{u}$*. Calculate the component of:*

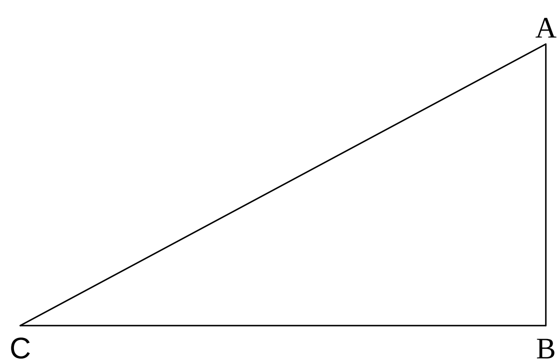
(a) $\mathbf{v} =< 2, 3, -5 >$ *in the direction of* $\mathbf{u} = \frac{1}{3} < 2, 2, 1 >$.

(b) $v =< a, b, c >$ *in the direction of* $u = \frac{1}{5} < 3, 0, 4 >$

8. *Show that the four points* $(1,-1,0)$, $(2,2,-2)$, $(5,3,1)$ *and* $(4,0,3)$ *are vertices of a rectangle. Is the rectangle a square?*

9. *Let* O *be the origin. Find a square* $OABC$ *then translate it by a suitable vector to get a square none of whose vertex is at the origin and none of its sides is parallel to a coordinate axis.*

10. *The three points* A, B *and* C *in space are such that angle* ABC *is a right angle.*



Let $\mathbf{u} = \vec{AB}$ *and* $\mathbf{v} = \vec{AC}$*. Show that the length of the side* AB *is* $\dfrac{\boldsymbol{v}\cdot\mathbf{u}}{||\mathbf{u}||} = \boldsymbol{v}\cdot\left(\dfrac{\mathbf{u}}{||\mathbf{u}||}\right)$*. In other words, the length of* AB *is the dot product of* $\mathbf{v}$ *and a unit vector parallel to* $\mathbf{u}$*.*