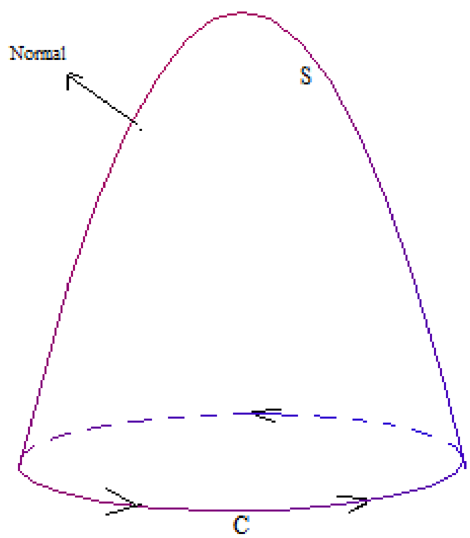
Stokes's Theorem

We earlier pointed out that if we introduce the vector field $\mathbf{F}(x,y,z) = M(x,y)\mathbf{i}+N(x,y)\mathbf{j}+0\mathbf{k}$, then Green's theorem $\int_C M(x,y)dx + N(x,y)dy = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$ in a plane may be stated in vector form as

$$\int_{\partial R} \mathbf{F}\cdot\vec{dl} = \iint\limits_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA.$$

Stokes theorem generalizes this form to a vector field $\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i}+F_2(x,y,z)\mathbf{j}+F_3(x,y,z)\mathbf{k}$ and a suitable surface S in three dimensional space. Like Green's theorem, it relates some integral over S to some line integral over the positively oriented curve C that forms the boundary of S. In Green's theorem, the boundary ∂R of R was declared positively oriented, (as the parameter defining it increases), one trace the curve in such a way that the interior of R is to one's left. In Stokes theorem, we have to spell out what it means for the boundary C of a given orientable surface S to be positively oriented. To this end, imagine pointing the thumb of your right hand towards a normal to S. If you curl your fingers, they will point in the positive direction of the boundary of S.



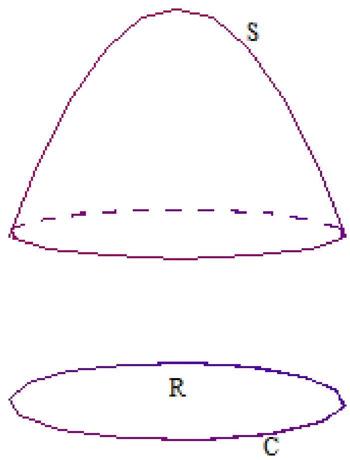
The direction of a positively oriented boundary C.

Theorem 1 *Let S be an orientable surface with unit normal $\mathbf{n}(x,y,z)$ at (x,y,z). Assume that its boundary ∂S is a simple closed positively oriented curve. Let $\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$ be a vector field whose components $F_1(x,y,z)$, $F_2(x,y,z)$, $F_3(x,y,z)$ have continuous partial derivatives on an open set containing S. Then*

$$\int_{\partial S} \mathbf{F}\cdot\vec{dl} = \iint\limits_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

Proof. *Assume that S is the graph of some function $g(x,y)$ defined on a set R in the x - y plane. Thus*

$S = \{(x, y, g(x, y) : (x, y) \in R\}$.



In particular, the boundary of S is the image of the curve C bounding R. We choose the unit normal at $(x, y, g(x, y))$ that has a positive $\mathbf{k}$ - component and it is

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}},$$

where the partial derivatives are evaluated at (x, y). From the definition of $\nabla \times \mathbf{F}$, it follows that

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = \frac{1}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}} \left[-\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\frac{\partial g}{\partial x} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\frac{\partial g}{\partial y} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

*Using formula (**??**), on page **??** we obtain*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_R \left[\left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\right)\frac{\partial g}{\partial x} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\frac{\partial g}{\partial y} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right] dA \qquad (1)$$

*Let the boundary C of R be the set $\{(x(t), y(t)) : a \le t \le b\}$. Then ∂S, the boundary of S, is the set $\{(x(t), y(t), g(x(t), y(t))) : a \le t \le b\}$. By Definition **??** on page **??**,*

$$\begin{aligned}\int_{\partial S} \mathbf{F} \cdot \vec{dl} &= \int_{\partial S} F_1(x, y, z)dx + F_2(x, y, z)dy + F_3(x, y, z)dz \\ &= \int_a^b \left(F_1(x(t), y(t), z(t))x'(t) + F_2(x(t), y(t), z(t))y'(t) + F_3(x(t), y(t), z(t))z'(t)\right) dt\end{aligned}$$

Since $z(t) = g(x(t), y(t))$, the chain rule gives

$$z'(t) = \frac{\partial g}{\partial x}x'(t) + \frac{\partial g}{\partial y}y'(t)$$

with the partial derivatives evaluated at $(x(t), y(t))$. This implies that

$$F_3(x(t), y(t), z(t))z'(t) = F_3(x(t), y(t), g(x(t), y(t)))\frac{\partial g}{\partial x}x'(t) + F_3(x(t), y(t), g(x(t), y(t)))\frac{\partial g}{\partial y}y'(t).$$

To save space we write $(x(t), y(t), g(x(t), y(t))$ as $(x, y, g(x, y))$. Therefore

$$\int_{\partial S} \mathbf{F}\cdot\vec{dl} = \int_a^b \left(\left[F_1(x,y,g(x,y)) + F_3(x,y,g(x,y))\frac{\partial g}{\partial x} \right] x'(t) + \left[F_2(x,y,g(x,y) + F_3(x,y,g(x,y))\frac{\partial g}{\partial y} \right] y'(t) \right) dt.$$

Now consider the functions $u(x, y)$ and $v(x, y)$ defined on R by

$$u(x,y) = F_1(x,y,g(x,y)) + F_3(x,y,g(x,y))\frac{\partial g}{\partial x}, \text{ and}$$

$$v(x,y) = F_2(x,y,g(x,y) + F_3(x,y,g(x,y))\frac{\partial g}{\partial y}$$

The definition of a line integral with respect to a variable implies that

$$\int_a^b \left[F_1(x(t),y(t),g(x(t),y(t))) + F_3(x(t),y(t),g(x(t),y(t)))\frac{\partial g}{\partial x} \right] x'(t) = \int_C u(x,y)dx$$

and

$$\int_a^b \left[F_2(x(t),y(t),g(x(t),y(t))) + F_3(x(t),y(t),g(x(t),y(t)))\frac{\partial g}{\partial y} \right] y'(t) = \int_C v(x,y)dy$$

Therefore

$$\int_{\partial S} \mathbf{F}\cdot\vec{dl} = \int_C u(x,y)dx + v(x,y)dy \tag{2}$$

Green's theorem may be applied to the right hand side of (2) to give

$$\int_{\partial S} \mathbf{F}\cdot\vec{dl} = \iint\limits_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA \tag{3}$$

We use the chain rule to evaluate the partial derivatives in (3):

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial}{\partial x}\left(F_2(x,y,g(x,y)) + F_3(x,y,g(x,y))\frac{\partial g}{\partial y} \right) \\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\frac{\partial g}{\partial x} + \left(\frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial z}\frac{\partial g}{\partial x} \right)\frac{\partial g}{\partial y} + F_3\frac{\partial^2 g}{\partial x \partial y} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(F_1(x,y,g(x,y)) + F_3(x,y,g(x,y))\frac{\partial g}{\partial x} \right) \\ &= \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial g}{\partial y} + \left(\frac{\partial F_3}{\partial y} + \frac{\partial F_3}{\partial z}\frac{\partial g}{\partial y} \right)\frac{\partial g}{\partial x} + F_3\frac{\partial^2 g}{\partial y \partial x} \end{aligned}$$

Now subtract to get

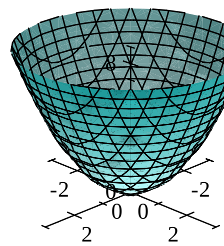
$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y} \right)\frac{\partial g}{\partial x} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right)\frac{\partial g}{\partial y} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \tag{4}$$

The right hand side of (4) is the integrand in (1). Therefore

$$\int_{\partial S} \mathbf{F}\cdot\vec{dl} = \iint\limits_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} dS.$$

■

Example 2 *Let S be the paraboloid $z = h(x,y) = x^2 + y^2$, $-3 \le x \le 3$, $-3 \le y \le 3$ and* $\mathbf{F}$ *be the vector field* $\mathbf{F}(x,y,z) = 2y\mathbf{i} + 3x\mathbf{j} + z^2\mathbf{k}$.



Clearly, ∂S is the circle $\{(3\cos t, 3\sin t, 9) : 0 \le t \le 2\pi\}$ and

$$\begin{aligned}\int_{\partial S} \mathbf{F}\cdot\vec{dl} &= \int_{\partial S} 2y dx + 3x dy + z^2 dz = \int_0^{2\pi} (-18\sin^2 t + 27\cos^2 t)dt \\ &= \int_0^{2\pi} \left[-9\left(1-\cos 2t\right) + 27\left(1+\cos 2t\right)\right] dt = 9\pi\end{aligned}$$

It turns out that $\nabla \times \mathbf{F} = \mathbf{k}$. A normal to S is $< -h_x, -h_y, 1 >=< -2x, -3y, 1 >$. Its norm is $\sqrt{4x^2+4y^2+1} = \sqrt{4z+1}$, therefore a unit normal to S is $\mathbf{n} = \frac{1}{\sqrt{4z+1}} < -2x, -3y, 1 >$. It follows that

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_S \frac{1}{\sqrt{4z+1}} dS = \iint_R \frac{\sqrt{h_x^2 + h_y^2 + 1}}{\sqrt{4z+1}} dA$$

where R is the disc$\{(x,y) : -3 \le x \le 3$ and $-3 \le y \le 3\}$. Since $\sqrt{h_x^2 + h_y^2 + 1} = \sqrt{4x^2+4y^2+1} = \sqrt{4z+1}$, the integral simplifies to

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_R \frac{\sqrt{4z+1}}{\sqrt{4z+1}} dA = \iint_R 1 dA = 9\pi.$$

This verifies Stokes's theorem.