Surface Integral of a Vector Field

To get an intuitive idea of the surface integral of a vector field, imagine a filter through which a certain fluid flows to be purified. The impurities are removed as the fluid crosses a surface S in the filter. Assume that the fluid velocity depends on position in space. Thus there is a function $\mathbf{F}(x, y, z) = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k}$ that gives the velocity of the fluid at each point (x, y, z). The volume of fluid that crosses S in unit time is called the flux of the vector field across S. It, (i.e. the flux), is zero if the fluid moves parallel to S. This suggests that at each point (x, y, z) of S we should consider the component of $\mathbf{F}$ that is normal to S at (x, y, z). That component is $\mathbf{F} \cdot \mathbf{n}$ where $\mathbf{n}$ is a unit normal to S at (x, y, z). Partition the surface into smaller elements S_{ij} with area $\triangle S_{ij}$. An estimate of the fluid that crosses S_{ij} in unit time is $(\mathbf{F} \cdot \mathbf{n})\,\triangle S_{ij}$. The sum

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(\mathbf{F} \cdot \mathbf{n})\,\triangle S_{ij} \tag{1}$$

should give an estimate of the flux of $\mathbf{F}$ across S and the limit of such sums as the areas $\triangle S_{ij}$ shrink to zero should be the exact value of the flux. That limit is denoted by

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS$$

and it is called the **flux integral of** F **across** S. In forming the sum (1), we assumed that a normal $\mathbf{n}$ is defined at every point of S. If it can be defined "continuously on S" then S is called an orientable surface. More precisely, a surface S is called an **orientable surface** if it is possible to assign a normal $\mathbf{n}(x, y, z)$ to each point (x, y, z) of S in such a way that when (x_1, y_1, z_1) is close to (x_2, y_2, z_2) then the two normals $\mathbf{n}(x_1, y_1, z_1)$ and $\mathbf{n}(x_2, y_2, z_2)$ are almost identical. That is; when $||(x_1, y_1, z_1) - (x_2, y_2, z_2)||$ is close to zero then $||\mathbf{n}(x_1, y_1, z_1) - \mathbf{n}(x_2, y_2, z_2)||$ is also close to zero.

Definition 1 *Let S be an orientable surface with normal $\mathbf{n}(x, y, z)$ at a point (x, y, z) of S. Let $\mathbf{F}(x, y, z)$ be a vector field defined on some open set containing S. Then the surface integral of $\mathbf{F}$ over S, (or the flux of $\mathbf{F}$ across S), is the number*

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS.$$

Example 2 *Let $\mathbf{F}(x, y, z) = xy\mathbf{i} + 3yz\mathbf{j} - xyz\mathbf{k}$ and S be the part of the plane $3x + 2y - 3z = 2$ with $-1 \le x \le 2$ and $0 \le y \le 1$. A unit normal to S at a point(x, y, z) is $\mathbf{n} = \frac{1}{\sqrt{22}} < 3, 2, -3 >$, therefore $\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{22}}(3xy + 6yz + 3xyz)$. It follows that*

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \frac{1}{\sqrt{22}}\iint_S (3xy + 6yz + 3xyz)\,dS.$$

*S is the surface $\left\{\left(x, y, \frac{3x+2y-2}{3}\right) : -1 \le x \le 2 \text{ and } 0 \le y \le 1\right\}$. Let $R = \{(x, y, z) : -1 \le x \le 2 \text{ and } 0 \le y \le 1\}$. By formula (**??**),*

$$\begin{aligned}
\iint_S (3xy + 6yz + 3xyz)\,dS &= \frac{1}{\sqrt{22}}\iint_R \left(3xy + (6y + 3xy)\left(\frac{3x + 2y - 2}{3}\right)\right)\left(\sqrt{1 + \frac{4}{9} + 1}\right) dA \\
&= \frac{1}{3}\int_{-1}^{4}\int_{0}^{5}\left(-4y + 4y^2 + 7xy + 3x^2y + 2xy^2\right) dydx \\
&= \frac{1}{3}\int_{-1}^{4}\left(-\frac{2}{3} + \frac{25x}{6} + \frac{3x^2}{2}\right) dx = \frac{1}{3}\left[-\frac{2x}{3} + \frac{25x^2}{12} + \frac{x^3}{2}\right]_{-1}^{2} = \frac{35}{12}
\end{aligned}$$

Example 3 *Let* $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} - z^2\mathbf{k}$ *and* S *be the hemisphere* $\left\{(x,y,z) : x^2+y^2+z^2 = 4 \text{ and } z \geq 0\right\}$. *Thus* S *is the graph of* $z(x,y) = \sqrt{4-x^2-y^2}$. *We choose the normal to the hemisphere* $(x,y,z(x,y))$ *that points outside the hemisphere. It must have a positive* z *- component, therefore we should take*

$$-\frac{\partial z}{\partial x}(x,y)\mathbf{i} - \frac{\partial z}{\partial y}(x,y)\mathbf{j} + \mathbf{k} = \frac{x\mathbf{i}}{\sqrt{4-x^2-y^2}} + \frac{y\mathbf{j}}{\sqrt{4-x^2-y^2}} + \mathbf{k} = \frac{x\mathbf{i}}{z} + \frac{y\mathbf{j}}{z} + \mathbf{k},$$

(see problem **??** *on page* **??***). Its norm is*

$$\sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1} = \sqrt{\frac{x^2+y^2+z^2}{z^2}} = \sqrt{\frac{4}{z^2}} = \frac{2}{z}$$

Therefore the corresponding unit normal is

$$\mathbf{n} = \frac{z}{2}\left(\frac{x\mathbf{i}}{z} + \frac{y\mathbf{j}}{z} + \mathbf{k}\right) = \frac{x\mathbf{i}}{2} + \frac{y\mathbf{j}}{2} + \frac{z\mathbf{k}}{2}$$

It follows that

$$\mathbf{F}\cdot\mathbf{n} = \frac{z}{2}\left(xy + xy - z^2\right) = \frac{z}{2}\left(2xy - z^2\right) = \frac{z}{2}\left(x^2+y^2+2xy-4\right) = \frac{z}{2}\left((x+y)^2 - 4\right)$$

Denote the circle $\left\{(x,y) : x^2+y^2 = 4\right\}$ *by* C *and the region it encloses by* R. *Then*

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_S \frac{z}{2}\left((x+y)^2-4\right)dS = \iint_R \frac{z}{2}\left((x+y)^2-4\right)\left(\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\right)dA \\ &= \iint_R \left((x+y)^2-4\right)dA\end{aligned}$$

This is easily evaluated when we change to polar coordinates:

$$\iint_R \left((x+y)^2-4\right)dA = \int_0^{2\pi}\int_0^2 \left((r\cos\theta + r\sin\theta)^2 - 4\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left(r^3(1+\sin 2\theta) - 4r\right)dr\,d\theta = -8\pi$$

Exercise 4 *Figure (i) is the solid enclosed by the paraboloid* $\left\{(x,y,z) : z = 16 - x^2 - y^2 \text{ and } z \geq 0\right\}$ *and the xy plane; Figure (ii) is the solid enclosed by the hemisphere* $\left\{(x,y,z) : x^2+y^2+z^2 = 4 \text{ and } z \geq 0\right\}$ *and the xy plane.*

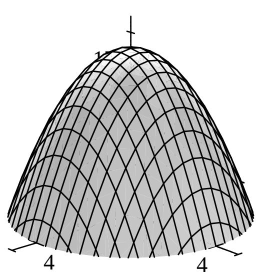


Figure (i)

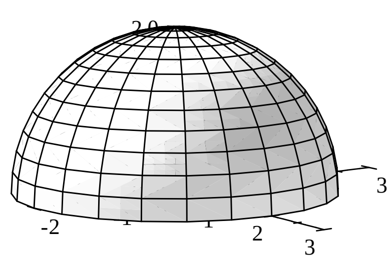


Figure (ii)

1. *Evaluate the flux of* $\mathbf{F}(x,y,z) = 3x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$ *across the surface* S *that encloses the solid in Figure (i).*
2. *Evaluate the flux of* $\mathbf{F}(x,y,z) = 3z\mathbf{i} - 2z\mathbf{j} + (x - y + 3)\,\mathbf{k}$ *across the surface* S *that encloses the solid in Figure (ii).*

The Divergence Theorem

In Remark **??** on page **??**, we pointed out that Green's theorem for a function defined in a plane may be given in the form

$$\int_{\partial R} \mathbf{F} \cdot \mathbf{n} dl = \iint_{R} \boldsymbol{\nabla} \cdot \mathbf{F} dA$$

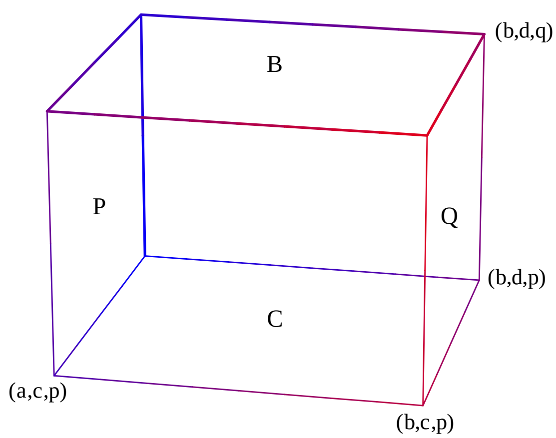
The divergence theorem is essentially an extension of this form to a function of three variables. It may be stated as follows:

Theorem 5 *Let* V *be a subset of* $\mathbb{R}^3$ *that is bounded by a closed orientable surface* S*. Let* $\mathbf{n}(x,y,z)$ *be the outward normal to* S *at* $(x,y,z) \in S$*. Let* $\mathbf{F} = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$ *be a given vector field whose components* F_1*,* F_2*, and* F_3 *have continuous partial derivatives in* V*. Then*

$$\iint_{S} \mathbf{F} \cdot \mathbf{n} dS = \iiint_{V} \boldsymbol{\nabla} \cdot \mathbf{F} dV$$

Proof. *Like Green's theorem, a proof of this statement for a general subset* V *of* $\mathbb{R}^3$ *requires some heavy duty tools we have not developed. We prove the special case when* V *is a rectangular box whose sides are parallel to the coordinate planes. Thus*

$$V = \{(x,y,z) : a \le x \le b,\ c \le y \le d \text{ and } p \le z \le q \text{ with } a,b,c,d,p,q \text{ constants}\}$$



Consider the top and bottom faces B *and* C *of the box. The outward normal to* B *is* $\mathbf{k}$ *whereas the outward normal to* C *is* $-\mathbf{k}$*. Therefore*

$$\begin{aligned}
\iint_{C} \mathbf{F} \cdot \mathbf{n} dA + \iint_{B} \mathbf{F} \cdot \mathbf{n} dA &= -\iint_{C} F_3 dA + \iint_{B} F_3 dA \\
&= -\int_a^b \int_c^d F_3(x,y,p) dy dx + \int_a^b \int_c^d F_3(x,y,q) dy dx \\
&= \int_a^b \int_c^d \left(F_3(x,y,q) - F_3(x,y,p)\right) dy dx
\end{aligned}$$

Now observe that $F_3(x,y,q) - F_3(x,y,p)$ *may be written as* $\int_p^q \frac{\partial F_3}{\partial z}dz$*. Therefore*

$$\iint_C \mathbf{F}\cdot\mathbf{n}dA + \iint_B \mathbf{F}\cdot\mathbf{n}dA = \int_a^b\int_c^d\int_p^q \frac{\partial F_3}{\partial z}dzdydx = \iiint_V \frac{\partial F_3}{\partial z}dV$$

Turning to the faces P *and* Q*, similar computations give*

$$\begin{aligned}\iint_P \mathbf{F}\cdot\mathbf{n}dA + \iint_Q \mathbf{F}\cdot\mathbf{n}dA &= -\int_a^b\int_p^q F_2(x,c,z)dzdx + \int_a^b\int_p^q F_2(x,d,z)dzdx \\ &= \int_a^b\int_p^q\int_c^d \frac{\partial F_2}{\partial y}dydzdx = \iiint_V \frac{\partial F_2}{\partial y}dV\end{aligned}$$

Denote the remaining pair of opposite faces by H *and* K*. Then*

$$\iint_H \mathbf{F}\cdot\mathbf{n}dA + \iint_K \mathbf{F}\cdot\mathbf{n}dA = \int_p^q\int_c^d\int_a^b \frac{\partial F_1}{\partial x}dxdydz = \iiint_V \frac{\partial F_1}{\partial x}dV$$

Since $\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iint_C \mathbf{F}\cdot\mathbf{n}dA + \iint_B \mathbf{F}\cdot\mathbf{n}dA + \iint_P \mathbf{F}\cdot\mathbf{n}dA + \iint_Q \mathbf{F}\cdot\mathbf{n}dA + \iint_H \mathbf{F}\cdot\mathbf{n}dA + \iint_K \mathbf{F}\cdot\mathbf{n}dA$*, it follows that*

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}dS &= \iiint_V \frac{\partial F_3}{\partial z}dV + \iiint_V \frac{\partial F_2}{\partial y}dV + \iiint_V \frac{\partial F_1}{\partial x}dV \\ &= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dV = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}dV\end{aligned}$$

A simple generalization of this is the case in which V *is a union* $V_1\cup\cdots\cup V_m$ *of a finite number of such boxes which intersect, if at all they do, only in their common parallel faces. Let their surfaces be* $S_1,\ldots,S_m$*. Since*

$$\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS + \cdots + \iint_{S_m} \mathbf{F}\cdot\mathbf{n}dS$$

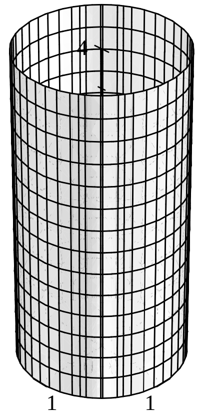
and

$$\iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}dV = \iiint_{V_1} \boldsymbol{\nabla}\cdot\mathbf{F}dV + \cdots + \iiint_{V_m} \boldsymbol{\nabla}\cdot\mathbf{F}dV$$

it follows that the theorem also holds for such sets V*. To prove the theorem in its general form, one has to show that a subset* V *of* $\mathbb{R}^3$ *with an orientable surface is a limit of such boxes and go on to deduce that the theorem also applies to it.* ■

Exercise 6

1. Let S be the cylinder $x^2 + y^2 = 1$, $0 \le z \le 4$.

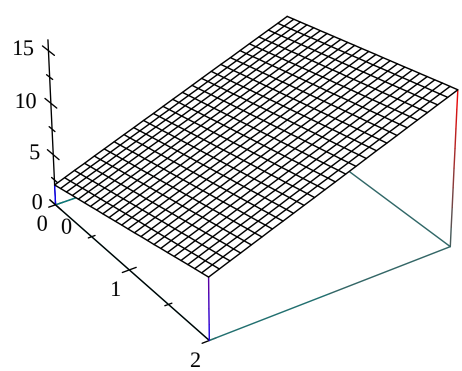


You are required to evaluate $\iint\limits_S \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F}(x, y, z) = yz\mathbf{i} + 4x^2\mathbf{j} + z^2\mathbf{k}$. Let V be the solid enclosed by the cylinder.

(a) Show that $\iiint\limits_V \nabla \cdot \mathbf{F} dV = 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{-\sqrt{1-x^2}}\int_0^4 z dz dy dx$.

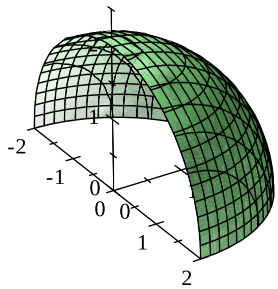
(b) Use the divergence theorem to evaluate $\iint\limits_S \mathbf{F} \cdot \mathbf{n} dS$.

2. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + yz\mathbf{j} - z(x + y)\mathbf{k}$ and S be the surface enclosing the solid between the plane $z = 2x + 3y + 2$ and the rectangle $\{(x, y, 0) : 0 \le x \le 2$ and $0 \le y \le 3\}$. Use the divergence theorem to evaluate the surface integral $\iint\limits_S \mathbf{F} \cdot \mathbf{n} dS$



3. The figure below shows the part of the hemisphere above the x - y plane and to the right of the x - z plane. Let V be the part of this solid that is above the $z = 1$ plane and S be the surface that encloses

V.



Use the divergence theorem to evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} dS$ *where* $\mathbf{F}(x, y, z) = y\mathbf{i} + 4x\mathbf{j} + xyz\mathbf{k}$.

4. *The figure below shows the region* V *enclosed by the plane* $z = x + y + 2$ *and the cylinder* $x^2 + y^2 = 1$, $0 \le z \le 4$. *Let* S *be the surface enclosing* V. *Use the divergence theorem to evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n} dS$ *where* $\mathbf{F}(x, y, z) = x\mathbf{i} + 4y\mathbf{j} + z\mathbf{k}$.