Line Integral of a Vector Field

The line integral of a vector field $\mathbf{F}(x,y,z) =< F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) >$ along a curve C defined by $\mathbf{c}(t) =< x(t), y(t), z(t) >$, $a \le t \le b$ is calculated the same way we calculated the work done by a variable force. The first step is to partition C into small elements, by dividing the interval $[a,b]$ into small subintervals $[t_0,t_1]$, $[t_1,t_2]$, $\ldots$, $[t_{n-1},t_n]$ where $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$. Let $\triangle x_i = x(t_{i+1}) - x(t_i)$, $\triangle y_i = y(t_{i+1}) - y(t_i)$, $\triangle z_i = z(t_{i+1}) - z(t_i)$ and $\overrightarrow{\triangle l_i} = \triangle x_i \mathbf{i} + \triangle y_i \mathbf{j} + \triangle z_i \mathbf{k}$. The next step is to form the sum

$$\sum_{i=0}^{n-1} \mathbf{F}\left(x(t_i), y(t_i), z(t_i)\right) \cdot \overrightarrow{\triangle l_i} = \sum_{i=0}^{n-1} \mathbf{F}\left(x(t_i), y(t_i), z(t_i)\right) \cdot \left(\left[x(t_{i+1}) - x(t_i)\right] \mathbf{i} + \left[y(t_{i+1}) - y(t_i)\right] \mathbf{j} + \left[z(t_{i+1}) - z(t_i)\right] \mathbf{k}\right) \tag{1}$$

The right hand side of (1) expands as

$$\sum_{i=0}^{n-1} F_1\left(x(t_i), y(t_i), z(t_i)\right)\left[x(t_{i+1}) - x(t_i)\right] + \sum_{i=0}^{n-1} F_2\left(x(t_i), y(t_i), z(t_i)\right)\left[y(t_{i+1}) - y(t_i)\right] + \sum_{i=0}^{n-1} F_3\left(x(t_i), y(t_i), z(t_i)\right)\left[z(t_{i+1}) - z(t_i)\right]$$

The limit of such sums as all the lengths $|t_{i+1} - t_i|$ tend to zero is called the line integral of $\mathbf{F}\left(x(t_i), y(t_i), z(t_i)\right)$ along C. It is denoted by $\int_C \mathbf{F}(x,y,z) \cdot \overrightarrow{dl}$. Actually we have already run into these limits. The limit of $\sum_{i=0}^{n-1} F_1\left(x(t_i), y(t_i), z(t_i)\right)\left[x(t_{i+1}) - x(t_i)\right]$ is the line integral of F_1 along C with respect to x, etc. Therefore

$$\int_C \mathbf{F}(x,y,z) \cdot \overrightarrow{dl} = \int_C F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz$$

If $x(t)$, $y(t)$, $z(t)$ have continuous derivatives then using the results we derived about line integrals with respect to variables gives

$$\int_C \mathbf{F}(x,y,z) \cdot \overrightarrow{dl} = \int_a^b \left(F_1(x(t), y(t), z(t))x'(t) + F_2(x(t), y(t), z(t))y'(t) + F_3(x(t), y(t), z(t))z'(t)\right) dt.$$

The following is the formal definition:

Definition 1 *Let $\mathbf{F}(x,y,z) =< F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) >$ be a given vector field and C be a curve parametrized by $\mathbf{c}(t) =< x(t), y(t), z(t) >$, $a \le t \le b$. The line integral of $\mathbf{F}$ along C is denoted by $\int_C \mathbf{F}(x,y,z) \cdot \overrightarrow{dl}$ and is defined by*

$$\int_C \mathbf{F}(x,y,z) \cdot \overrightarrow{dl} = \int_C F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz$$

Example 2 *Let $\mathbf{F}(x,y,z) =< yz, xz, xy >$ and C be defined by $\mathbf{c}(t) =< 4t+1, t^2, 2t-3 >$, $0 \le t \le 3$. Then $F_1(x,y,z) = yz$, $F_2(x,y,z) = xz$, $F_3(x,y,z) = xy$, $x(t) = 4t+1$, $y(y) = t^2$ and $z(t) = 2t-3$. The line integral of $\mathbf{F}$ over C is*

$$\begin{aligned} \int_C \mathbf{F}(x,y,z) \cdot \overrightarrow{dl} &= \int_C yzdx + xzdy + xydz \\ &= \int_0^3 \left[t^2 (2t-3) + (4t+1)(2t-3) 2t + t^2 (4t+1) 2\right] dt \\ &= \int_0^3 \left(32t^3 - 30t^2 - 6t\right) dt \\ &= \left[4t^4 - 10t^3 - 3t^2\right]_0^3 = 27 \end{aligned}$$

Exercise 3 *Evaluate the line integral* $\int_C \mathbf{F}(x,y,z)\cdot \vec{dl}$ *given that*

1. $\mathbf{F}(x,y,z) = 3y\mathbf{i} - 4z\mathbf{j} - 2xyz\mathbf{k}$ *and* C *is defined by* $\mathbf{c}(t) = < t^2, t^3, 2t >, 0 \le t \le 1$.
2. $\mathbf{F}(x,y,z) = y\mathbf{i} - x\mathbf{j} - (z+y)\,\mathbf{k}$ *and* C *is defined by* $\mathbf{c}(t) = < \cos t, \sin t, t >, 0 \le t \le \pi/3$.
3. $\mathbf{F}(x,y,z) = (x+y)\,\mathbf{i} - xy\mathbf{j} - (z+y)\,\mathbf{k}$ *and* C *is defined by* $\mathbf{c}(t) = < e^t, 3e^t, t >, 0 \le t \le \pi/3$.

The next theorem gives us a quick method of evaluating a line integral of a conservative vector field.

Theorem 4 *Let* $\mathbf{F}(x,y,z) = < F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) >$ *be a conservative vector field with potential function* ϕ *and* C *be a smooth curve defined by* $\mathbf{c}(t) = < x(t), y(t), z(t) >$, $a \le t \le b$. *Then* $\int_C \mathbf{F}(x,y,z)\cdot \vec{dl} = \boldsymbol{\phi}(x(b),y(b),z(b)) - \boldsymbol{\phi}(x(a),y(a),z(a))$.

Proof. *By definition,* $\int_C \mathbf{F}(x,y,z)\cdot \vec{dl} = \int_C F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz$. *The potential function satisfies the three conditions*

$$F_1(x,y,z) = \phi_x(x,y,z), \quad F_2(x,y,z) = \phi_y(x,y,z), \quad \text{and} \quad F_3(x,y,z) = \phi_z(x,y,z).$$

Therefore

$$\begin{aligned}\int_C \mathbf{F}(x,y,z)\cdot \vec{dl} &= \int_C \phi_x(x,y,z)dx + \phi_y(x,y,z)dy + \phi_z(x,y,z)dz \\ &= \int_a^b \left[\phi_x(x(t),y(t),z(t))x'(t) + \phi_y(x(t),y(t),z(t))y'(t) + \phi_z(x(t),y(t),z(t))z'(t)\right] dt\end{aligned}$$

By the chain rule, the real-valued function $g(t) = \phi \circ \mathbf{c}(t)$ *of one variable* t *has derivative* $g'(t)$ *given by*

$$g'(t) = \begin{pmatrix} \phi_x(x(t),y(t),z(t)) & \phi_y(x(t),y(t),z(t)) & \phi_z(x(t),y(t),z(t)) \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$$

On multiplying the two matrices we get the very integrand in the above integral. Therefore

$$\int_C \mathbf{F}(x,y,z)\cdot \vec{dl} = \int_a^b g'(t)dt = g(b) - g(a) = \boldsymbol{\phi}(x(b),y(b),z(b)) - \boldsymbol{\phi}(x(a),y(a),z(a)).$$

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Two useful results follow from Theorem 4

1. If $\mathbf{F}(x,y,z)$ is a conservative vector field and C is a smooth closed curve then $\int_C \mathbf{F}(x,y,z)\cdot \vec{dl} = \mathbf{0}$. To see this take a potential ϕ for F. Say C starts and ends at a. Then

$$\int_C \mathbf{F}(x,y,z)\cdot \vec{dl} = \boldsymbol{\phi}(x(a),y(a),z(a)) - \boldsymbol{\phi}(x(a),y(a),z(a)) = 0.$$

2. If $\mathbf{F}(x,y,z)$ is a conservative vector field and C_1, C_2 are smooth curves that join two points P and Q then

$$\int_{C_1} \mathbf{F}(x,y,z)\cdot \vec{dl} = \int_{C_2} \mathbf{F}(x,y,z)\cdot \vec{dl}. \tag{2}$$

To see this, consider the piecewise smooth curve $C = C_1 - C_2$. It is closed, therefore

$$0 = \int_{C_1 - C_2} \mathbf{F}(x,y,z)\cdot \vec{dl} = \int_{C_1} \mathbf{F}(x,y,z)\cdot \vec{dl} - \int_{C_2} \mathbf{F}(x,y,z)\cdot \vec{dl}$$

which implies (2).

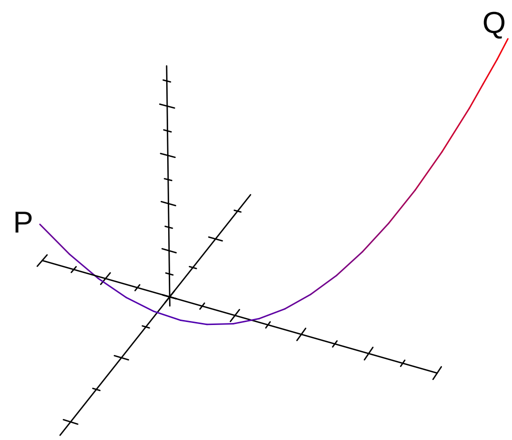
The converse of the assertion in (2) is also true under certain conditions, namely:

Theorem 5 *Let $\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$ be a vector field whose components $F_1(x,y,z)$, $F_2(x,y,z)$, and $F_3(x,y,z)$ have continuous first order partial derivatives. If $\int_{C_1} \mathbf{F}(x,y,z) \cdot \vec{dl} = \int_{C_2} \mathbf{F}(x,y,z) \cdot \vec{dl}$ for every pair of smooth curves C_1 and C_2 joining any two given points P and Q then $\mathbf{F}(x,y,z) = \boldsymbol{\nabla}\phi$ for some function ϕ.*

Proof. *We give a formula for ϕ then verify that it has the required properties. To this end, fix a point $P\,(a,b,c)$ in $\mathbb{R}^3$. Let $Q\,(x,y,z)$ be any point in $\mathbb{R}^3$ and C any smooth curve joining P and Q. We define*

$$\phi(x,y,z) = \int_C \mathbf{F}(x,y,z) \cdot \vec{dl}$$

This is a well-defined function because the integral is independent of the curve joining P and Q.



We must show that $\phi_x\,(x_0,y_0,z_0) = F_1\,(x_0,y_0,z_0)$, $\phi_y\,(x_0,y_0,z_0) = F_2\,(x_0,y_0,z_0)$, and $\phi_z\,(x_0,y_0,z_0) = F_3\,(x_0,y_0,z_0)$ for all (x_0,y_0,z_0) in the domain of $\mathbf{F}$. It suffices to verify any one of them because the same method applies to any one of the three. Therefore we show that

$$\lim_{h\to 0} \frac{\phi\,(x_0+h,y_0,z_0) - \phi\,(x_0+h,y_0,z_0)}{h} = F_1\,(x_0,y_0,z_0)$$

This is equivalent to

$$\lim_{h\to 0} \left| \frac{\phi\,(x_0+h,y_0,z_0) - \phi\,(x_0+h,y_0,z_0) - hF_1\,(x_0,y_0,z_0)}{h} \right| = 0$$

More precisely, we show that given any $\varepsilon > 0$, we can find a $\delta > 0$ such that $|h| < \delta$ implies

$$\left| \frac{\phi\,(x_0+h,y_0,z_0) - \phi\,(x_0+h,y_0,z_0) - hF_1\,(x_0,y_0,z_0)}{h} \right| < \varepsilon.$$

For any h, let R be the point (x_0+h,y_0,z_0) and C_1 the line segment joining (x_0,y_0,z_0) and (x_0+h,y_0,z_0). Then

$$\phi\,(x_0+h,y_0,z_0) - \phi\,(x_0+h,y_0,z_0) = \int_{C_1} \mathbf{F}(x,y,z) \cdot \vec{dl} \tag{3}$$

When we parametrize C_1 as $(x(t),y(t),z(t))$ where $x(t) = x_0 + th$, $y(t) = y_0$ and $z(t) = z_0$, the right hand side of (3) becomes

$$\int_0^1 F_1\,(x(t),y(t),z(t))\,x'(t)dt + \int_0^1 F_2\,(x(t),y(t),z(t))\,y'(t)dt + \int_0^1 F_3\,(x(t),y(t),z(t))\,z'(t)dt$$

Since $y\prime(t) = 0 = z\prime(t)$, we obtain

$$\phi(x_0 + h, y_0, z_0) - \phi(x_0 + h, y_0, z_0) = \int_0^1 F_1(x_0 + th, y_0, z_0) h dt.$$

Therefore

$$\left| \frac{\phi(x_0 + h, y_0, z_0) - \phi(x_0 + h, y_0, z_0) - hF_1(x_0, y_0, z_0)}{h} \right| = \left| \int_0^1 F_1(x_0 + th, y_0, z_0) dt - F_1(x_0, y_0, z_0) \right|$$

We may write $\int_0^1 F_1(x_0 + th, y_0, z_0) dt - F_1(x_0, y_0, z_0)$ as $\int_0^1 [F_1(x_0 + th, y_0, z_0) - F_1(x_0, y_0, z_0)]\, dt$. Now note that

$$\left| \int_0^1 [F_1(x_0 + th, y_0, z_0) - F_1(x_0, y_0, z_0)]\, dt \right| \leq \int_0^1 |F_1(x_0 + th, y_0, z_0) - F_1(x_0, y_0, z_0)|\, dt$$

By the Mean Value Theorem, there is a point (θ, y_0, z_0) on C_1 such that

$$|F_1(x_0 + th, y_0, z_0) - F_1(x_0, y_0, z_0)| = (|th|)\, |(F_1)_x(\theta, y_0, z_0)|\,.$$

The continuity of the partial derivatives of F_1 implies that there is a constant K and a positive number r such that $|(F_1)_x(x, y, z)| \leq K$ for all points (x, y, z) in the sphere centered at (x_0, y_0, z_0) with radius r. If $(x_0 + h, y_0, z_0)$ is such a point then

$$\left| \frac{\phi(x_0 + h, y_0, z_0) - \phi(x_0 + h, y_0, z_0) - hF_1(x_0, y_0, z_0)}{h} \right| < \int_0^1 K\, |th|\, dt = \tfrac{1}{2} K\, |h|$$

It follows that given $\varepsilon > 0$, if we choose any δ less than the smaller of the two numbers $\frac{2\varepsilon}{K}$ and r then $|h| < \delta$ implies

$$\left| \frac{\phi(x_0 + h, y_0, z_0) - \phi(x_0 + h, y_0, z_0) - hF_1(x_0, y_0, z_0)}{h} \right| < \varepsilon$$

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