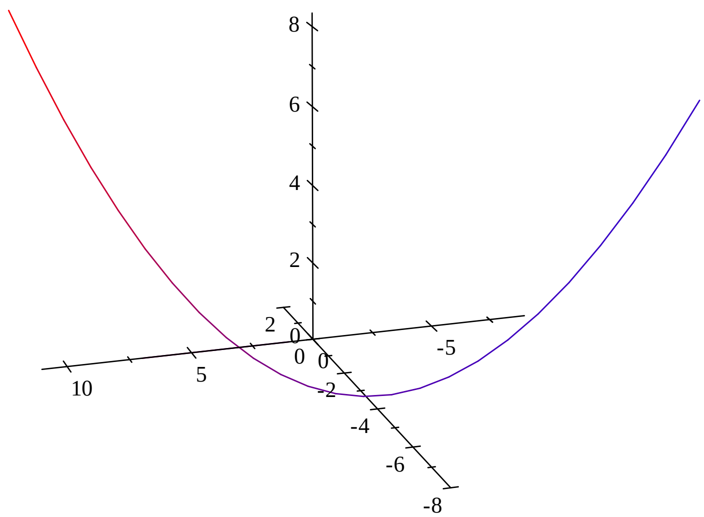
Line Integral of a Function With Respect to a Variable

Let $f(x, y, z)$ be a given function of the three variables x, y and z. We wish to introduce line integrals of f with respect to x or y or z. We will meet these integrals when integrating vector valued functions along curves. They are best illustrated by the concept of work done when a body, acted on by a variable force, is moved along a curve.

Example 1 *Consider a body that is acted on by a force* $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$, *when it is at a point* (x, y, z) *in space, and a curve* C *defined by* $\mathbf{c}(t) =< x(t), y(t), z(t) >$, $a \le t \le b$. *For a specific curve* C, *we take* $\mathbf{c}(t) =< t-3, 2t+1, \frac{1}{3}t^2 >$, $-5 \le t \le 5$.



The curve $\mathbf{c}(t) =< t-3, 2t+1, \frac{1}{3}t^2 >$, $-5 \le t \le 5$.

What is the work done to move the body along C *from a point* $(x(a), y(a), z(a))$ *to* $(x(b), y(b), z(b))$? *If* $\mathbf{F}(x, y, z)$ *is not a constant, we have to resort to integration for an answer. You should expect the usual procedure, which is:*

- *Partition the curve into smaller curved segments by partitioning the interval* $[a, b]$ *into smaller subintervals* $[t_0, t_1]$, $[t_1, t_2]$, ..., $[t_{n-1}, t_n]$ *where*

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

- *Estimate the work done to move the object along each curved segment.*
- *Total up to get an approximate value of the work done to move the object along the entire curve then determine the limit of such sums.*

  To get down to the details, denote the segment of the curve that starts from $(x(t_i), y(t_i), z(t_i))$ *and ends at* $(x(t_{i+1}), y(t_{i+1}), z(t_{i+1}))$ *by* C_i. *When the subinterval* $[t_i, t_{i+1}]$ *is sufficiently small, the force* $\mathbf{F}(x, y, z)$ *is practically constant along* C_i *and equal to*

$$\mathbf{F}\left(x(t_i), y(t_i), z(t_i)\right) = \left(F_1\left(x(t_i), y(t_i), z(t_i)\right), F_2\left(x(t_i), y(t_i), z(t_i)\right), F_3\left(x(t_i), y(t_i), z(t_i)\right)\right)$$

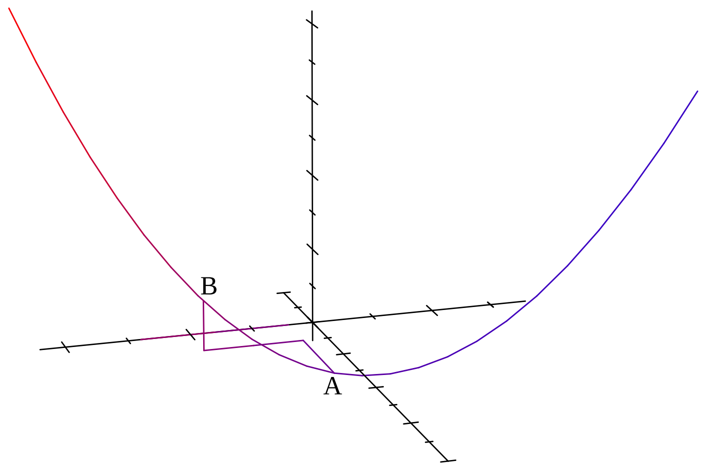
  We approximate the work done to move the object along C_i *by the work done to move it in 3 steps as follows:*

  1. *Move it from* $(x(t_i), y(t_i), z(t_i))$ *to* $(x(t_{i+1}), y(t_i), z(t_i))$ *along the line segment joining the two points.*

2. *Move it from* $(x(t_{i+1}), y(t_i), z(t_i))$ *to* $(x(t_{i+1}), y(t_{i+1}), z(t_i))$ *along the line segment joining the two points.*

3. *Move it from* $(x(t_{i+1}), y(t_{i+1}), z(t_i))$ *to* $(x(t_{i+1}), y(t_{i+1}), z(t_{i+1}))$ *along the line segment joining the two points.*

For example, if C_i *is the segment joining* $A(3,-1,0)$ *to* $B(1,-5,1.5)$ *then then we would move the object as shown below, from* $(3,-1,0)$ *to* $(1,-1,0)$*, then to* $(1,-5,0)$ *and finally to* $B\,(1,-5,1.5)$.



By definition, the work done by a force is

$$\text{(Magnitude of the force)} \times \text{(Distance moved in the direction of the force)}$$

In moving the object from $(x(t_i), y(t_i), z(t_i))$ *to* $(x(t_{i+1}), y(t_i), z(t_i))$ *along the line segment joining the two points, the distance moved in the direction of* $F_1\,(x(t_i), y(t_i), z(t_i))\,\mathbf{i}$ *is* $x(t_{i+1}) - x(t_i)$. *The distance is zero in the directions of* $F_2\,(x(t_i), y(t_i), z(t_i))\,\mathbf{j}$ *and* $F_3\,(x(t_i), y(t_i), z(t_i))\,\mathbf{k}$. *Therefore the work done is approximately*

$$F_1\,(x(t_i), y(t_i), z(t_i)) \times [x(t_{i+1}) - x(t_i)]$$

In moving the object from $(x(t_{i+1}), y(t_i), z(t_i))$ *to* $(x(t_{i+1}), y(t_{i+1}), z(t_i))$ *along the line segment joining the two points, the distance moved in the direction of* $F_2\,(x(t_i), y(t_i), z(t_i))\,\mathbf{j}$ *is* $y(t_{i+1}) - y(t_i)$, *and it is zero in the directions of* $F_1\,(x(t_i), y(t_i), z(t_i))\,\mathbf{i}$ *and* $F_3\,(x(t_i), y(t_i), z(t_i))\,\mathbf{k}$. *Therefore the work done is approximately*

$$F_2\,(x(t_i), y(t_i), z(t_i)) \times [y(t_{i+1}) - y(t_i)]$$

Finally, the work done in moving it from $(x(t_{i+1}), y(t_{i+1}), z(t_i))$ *to* $(x(t_{i+1}), y(t_{i+1}), z(t_{i+1}))$ *along the line segment joining the two points is approximately*

$$F_3\,(x(t_i), y(t_i), z(t_i)) \times [z(t_{i+1}) - z(t_i)]$$

Therefore an estimate of the work done to move the object from $(x(t_i), y(t_i), z(t_i))$ *to* $(x(t_{i+1}), y(t_{i+1}), z(t_{i+1}))$ *is*

$$\left.\begin{aligned} &F_1\,(x(t_i), y(t_i), z(t_i))\,[x(t_{i+1}) - x(t_i)] + F_2\,(x(t_i), y(t_i), z(t_i))\,[y(t_{i+1}) - y(t_i)] \\ &+F_3\,(x(t_i), y(t_i), z(t_i))\,[z(t_{i+1}) - z(t_i)] \end{aligned}\right\} \quad (1)$$

The sum of these n *approximations gives an approximate value of the work done to move the object from* $(x(-\pi), y(-\pi), z(-\pi))$ *to* $(x(\pi), y(\pi), z(\pi))$ *and it is*

$$\left.\begin{aligned} &\sum_{i=0}^{n-1} F_1\,(x(t_i), y(t_i), z(t_i))\,[x(t_{i+1}) - x(t_i)] + \sum_{i=0}^{n-1} F_2\,(x(t_i), y(t_i), z(t_i))\,[y(t_{i+1}) - y(t_i)] \\ &+\sum_{i=0}^{n-1} F_3\,(x(t_i), y(t_i), z(t_i))\,[z(t_{i+1}) - z(t_i)] \end{aligned}\right\} \quad (2)$$

The limit of these sums as all the lengths $[x(t_{i+1}) - x(t_i)]$, $[y(t_{i+1}) - y(t_i)]$, $[z(t_{i+1}) - z(t_i)]$ *shrink to zero should be the exact amount of work done to move the object from* $(x(-\pi), y(-\pi), z(-\pi))$ *to* $(x(\pi), y(\pi), z(\pi))$. *It is actually a sum of three limits which are:*

- *The limit of* $\sum_{i=0}^{n-1} F_1(x(t_i), y(t_i), z(t_i))[x(t_{i+1}) - x(t_i)]$ *as all the lengths* $[x(t_{i+1}) - x(t_i)]$ *shrink to zero. It is called the line integral of* F_1 *along* C *with respect to* x *and is denoted by*

$$\int_C F_1(x, y, z)dx$$

Why $\int_C F_1(x, y, z)dx$ *? Because we pick points* $(x(t_i), y(t_i), z(t_i))$ *on the curve* C, *multiply each function value* $F_1(x(t_i), y(t_i), z(t_i))$ *by the "length"* $[x(t_{i+1}) - x(t_i)]$ *involving* x *only then determine the limit of the sums*

$$\sum_{i=0}^{n-1} F_1(x(t_i), y(t_i), z(t_i))[x(t_{i+1}) - x(t_i)]$$

as the lengths of all the intervals $[t_0, t_1]$, $[t_1, t_2]$, $\ldots$, $[t_{n-1}, t_n]$ *shrink to zero.*

- *The limit of* $\sum_{i=0}^{n-1} F_2(x(t_i), y(t_i), z(t_i))[y(t_{i+1}) - y(t_i)]$ *as all the lengths* $[y(t_{i+1}) - y(t_i)]$ *shrink to zero. It is called the line integral of* F_2 *along* C *with respect to* y *and is denoted by*

$$\int_C F_2(x, y, z)dy$$

- *The limit of* $\sum_{i=0}^{n-1} F_3(x(t_i), y(t_i), z(t_i))[z(t_{i+1}) - z(t_i)]$ *as all the lengths* $[z(t_{i+1}) - z(t_i)]$ *shrink to zero. It is called the line integral of* F_3 *along* C *with respect to* z *and is denoted by*

$$\int_C F_3(x, y, z)dz$$

Therefore the work done to move the object from $(x(a), y(a), z(a))$ *to* $(x(b), y(b), z(b))$ *is*

$$\int_C F_1(x, y, z)dx + \int_C F_2(x, y, z)dy + \int_C F_3(x, y, z)dz$$

This is written briefly as

$$\int_C F_1(x, y, z)dx + F_2(x, y, z)dy + F_3(x, y, z)dz$$

To evaluate these integrals using antiderivatives, we use the Mean Value Theorem to replace the lengths $[x(t_{i+1}) - x(t_i)]$, $[y(t_{i+1}) - y(t_i)]$ *and* $[z(t_{i+1}) - z(t_i)]$ *with terms involving* $(t_{i+1} - t_i)$. *The theorem asserts that there is a number* θ_i *between* t_i *and* t_{i+1} *such that*

$$x(t_{i+1}) - x(t_i) = x'(\theta_i)(t_{i+1} - t_i)$$

If we assume that x' *is a continuous function of* t *then* $x'(\theta_i) \simeq x'(t_i)$, *hence* $x(t_{i+1}) - x(t_i)$ *may be approximated by* $x'(t_i)(t_{i+1} - t_i)$ *and* $\sum_{i=0}^{n-1} F_1(x(t_i), y(t_i), z(t_i))[x(t_{i+1}) - x(t_i)]$ *may be approximated by*

$$\sum_{i=0}^{n-1} F_1(x(t_i), y(t_i), z(t_i))\, x'(t_i)(t_{i+1} - t_i) \tag{3}$$

The limit of (3) as the lengths of all the intervals $[t_0, t_1]$, $[t_1, t_2]$, $\ldots$, $[t_{n-1}, t_n]$ *shrink to zero is*

$$\int_{-\pi}^{\pi} F_1(x(t), y(t), z(t))x'(t)dt$$

In other words, if $x'(t)$ *is continuous as a function of* t *then*

$$\int_C F_1(x, y, z)dx = \int_{-\pi}^{\pi} F_1(x(t), y(t), z(t))x'(t)dt$$

One shows in a similar way that if $y'(t)$ *and* $z'(t)$ *are continuous functions of* t *then*

$$\int_C F_2(x, y, z)dy = \int_{-\pi}^{\pi} F_2(x(t), y(t), z(t))y'(t)dt$$

and

$$\int_C F_3(x, y, z)dz = \int_{-\pi}^{\pi} F_3(x(t), y(t), z(t))z'(t)dt$$

Therefore the work done to move the object from $\mathbf{c}(a)$ *to* $\mathbf{c}(b)$ *is*

$$\int_{-a}^{b} [F_1(x(t), y(t), z(t))x'(t) + F_2(x(t), y(t), z(t))y'(t) + F_3(x(t), y(t), z(t))z'(t)]dt$$

In the specific example we cited, $x(t) = t - 3$, $y(t) = 2t + 1$, *and* $z(t) = \frac{1}{3}t^2$, $-5 \leq t \leq 5$. *Suppose*

$$F_1(x, y, z) = y^2,\ F_2(x, y, z) = 2y - x,\ \text{and } F_3(x, y, z) = 3x - z$$

Then $F_1(x(t), y(t), z(t))x'(t)dt = [y(t)]^2\,(1)\,dt = (2t+1)^2\,dt$, *and the integral of* F_1 *along* C *with respect to* x *is*

$$\int_{-5}^{5} (2t+1)^2\,dt = \frac{1}{6}\left[(2t+1)^3\ \right]_{-5}^{5} = \frac{1030}{3}$$

The integral of F_2 *along* C *with respect to* y *is*

$$\int_{-5}^{5} (2\,(2t+1) - t + 3)\,2dt = \int_{-5}^{5} (6t + 10)\,dt = \left[3t^2 + 10t\ \right]_{-5}^{5} = 100$$

The integral of F_3 *along* C *with respect to* z *is*

$$\int_{-5}^{5} \left(3\,(t-3) - \frac{1}{3}t^2\right)\frac{2t}{3}dt = \int_{-5}^{5} \left(2t^2 - 6t - \frac{2t^3}{9}\right)dt = \left[\frac{2t^3}{3} - 3t^2 - \frac{t^4}{18}\right]_{-5}^{5} = \frac{500}{3}$$

Therefore the work done to move the object is

$$\int_{-5}^{5}\left[(2t+1)^2 + (6t+10) + \left(2t^2 - 6t - \frac{2t^3}{9}\right)\right]dt = \frac{1030}{3} + 100 + \frac{500}{3} = 610 \text{ units.}$$

Remark 2 *If we denote* $x(t_{i+1}) - x\,(t_i)$ *by* $\triangle x_i$, $y(t_{i+1}) - y\,(t_i)$ *by* $\triangle y_i$, $z(t_{i+1}) - z\,(t_i)$ *by* $\triangle z_i$ *and define* $\overrightarrow{\triangle l_i} = \triangle x_i\mathbf{i} + \triangle y_i\mathbf{j} + \triangle z_i\mathbf{k}$ *then (1), (the work done to move the body from* $(x(t_i), y(t_i), z(t_i))$ *to* $(x(t_{i+1}), y(t_{i+1}), z(t_{i+1}))$*) is approximately equal to.*

$$\mathbf{F}(x_i, y_i, z_i) \cdot \overrightarrow{\triangle l_i}$$

Also (2), (the total work done to move the object from $(x(a), y(a), z(a))$ *to* $(x(b), y(b), z(b))$*) is approximately equal to*

$$\sum_{i=0}^{n-1} \mathbf{F}(x_i, y_i, z_i) \cdot \overrightarrow{\triangle l_i}$$

We will use these expressions to define line integrals of more general vector valued functions.

Example 3 *Let* $f(x,y) = x^3y^2 + 4xy - 1$ *and* C *be that part of the parabola* $y = x^2$ *on the interval* $[-1, 2]$. *To determine* $\int_C f(x,y)dx$ *and* $\int_C f(x,y)dy$.

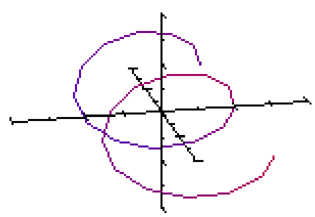
We must parametrize C. *One possible parametrization is* $x(t) = t$, $y(t) = t^2$, $-1 \le t \le 2$. *This implies that* $x'(t) = 1$ *and* $y'(t) = 2t$. *Therefore*

$$\begin{aligned}\int_C f(x,y)dx &= \int_{-1}^{2} \left(t^3 \left(t^2\right)^2 + 4t\left(t^2\right) - 1\right) dt = \int_{-1}^{2} \left(t^7 + 4t^3 - 1\right) dt \\ &= \left[\frac{t^8}{8} + t^4 - t\right]_{-1}^{2} = 43\tfrac{7}{8}\end{aligned}$$

and

$$\int_C f(x,y)dy = \int_{-1}^{2} \left(t^7 + 4t^3 - 1\right) 2tdt = \int_{-1}^{2} \left(2t^8 + 8t^4 - 2t\right) dt = 163.8$$

Example 4 *To evaluate* $\int_C \left(3x - z + 4y^2\right) dz$ *given that* C *is the curve* $\mathbf{c}(t) =< t, \cos \pi t, \sin \pi t >$, $-\frac{1}{2} \le t \le 1$, *(a helix).*



The helix $\mathbf{c}(t) =< t, \cos \pi t, \sin \pi t >$, $-\frac{1}{2} \le t \le 1$.

We are given that $x(t) = t$, $y(t) = \cos \pi t$ *and* $z(t) = \sin \pi t$. *Consequently,* $z'(t) = \pi \cos \pi t$. *The function to integrate with respect to* z *is* $f(x,y,z) = 3x - z + 4y^2$ *and*

$$f(x(t), y(t), z(t)) = 3t - \sin \pi t + 4\cos^2 \pi t$$

Therefore

$$\begin{aligned}\int_C \left(3x - z + 4y^2\right) dz &= \int_{-\frac{1}{2}}^{1} \left(3t - \sin \pi t + 4\cos^2 \pi t\right)(\pi \cos \pi t)\, dt \\ &= 3\pi \int_{-\frac{1}{2}}^{1} t \cos \pi t dt - \pi \int_{-\frac{1}{2}}^{1} \sin \pi t \cos \pi t dt + 4\pi \int_{-\frac{1}{2}}^{1} \cos^3 \pi t dt\end{aligned}$$

Integrating by parts gives

$$3\pi \int_{-\frac{1}{2}}^{1} t \cos \pi t dt = 3\pi \left[\frac{t \sin \pi t}{\pi} + \frac{\cos \pi t}{\pi^2}\right]_{-\frac{1}{2}}^{1} = -\frac{3}{\pi} - \frac{3}{2}$$

The second integral is

$$-\pi \int_{-\frac{1}{2}}^{1} \sin \pi t \cos \pi t dt = -\pi \left[\frac{\sin^2 \pi t}{2\pi}\right]_{-\frac{1}{2}}^{1} = -\pi\left(-\frac{1}{2\pi}\right) = \frac{1}{2}$$

The last integral is

$$\begin{aligned} 4\pi\int_{-\frac{1}{2}}^{1} \cos^3 \pi t dt &= 4\pi\int_{-\frac{1}{2}}^{1} \left(1-\sin^2 \pi t\right)\cos \pi t dt = 4\pi\int_{-\frac{1}{2}}^{1} \left(\cos \pi t - \sin^2 \pi t \cos \pi t\right) dt \\ &= 4\pi\left[\frac{\sin \pi t}{\pi} - \frac{\sin^3 \pi t}{3\pi}\right]_{-\frac{1}{2}}^{1} = 0 - 4\left(-1+\frac{1}{3}\right) = \frac{8}{3} \end{aligned}$$

Therefore

$$\int_C \left(3x - z + 4y^2\right) dz = -\frac{3}{\pi} - \frac{3}{2} + \frac{1}{2} + \frac{8}{3} = \frac{5\pi - 9}{3\pi}$$

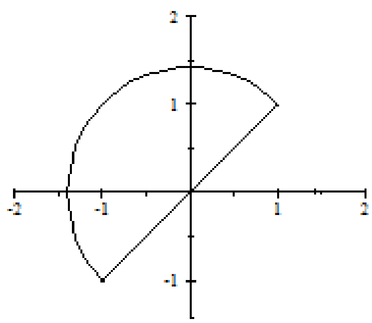
Definition 5 *If $C = C_1 + C_2 + \cdots C_k$ then we define $\int_C f(x,y)dx$ and $\int_C f(x,y)dy$ by*

$$\int_C f(x,y)dx = \int_{C_1} f(x,y)dx + \int_{C_2} f(x,y)dx + \cdots + \int_{C_k} f(x,y)dx$$

and

$$\int_C f(x,y)dy = \int_{C_1} f(x,y)dy + \int_{C_2} f(x,y)dy + \cdots + \int_{C_k} f(x,y)dy$$

Example 6 *To determine $\int_C f(x,y)dy$ where $f(x,y) = x^2 + y^2 - x$ and C is the closed curve, shown below, consisting of the line segment from $(-1,-1)$ to $(1,1)$ and the part of the circle $x^2 + y^2 = 2$ above the line $y = x$.*



We start by parametrizing C. It is the sum of the line segment C_1 and the segment C_2 of the circle $x^2+y^2 = 2$. Clearly $C_1 = \{(x(t), y(t))\}$ where $x(t) = t$, $y(t) = t$ and $-1 \leq t \leq 1$. The semicircle is the set $\{(x(t), y(t))\}$ with $x(t) = \cos t$, $y(t) = \sin t$ and $\frac{\pi}{4} \leq t \leq \frac{5\pi}{4}$. It follows that

$$\int_{C_1} f(x,y)dy = \int_{-1}^{1} \left(t^2 + t^2 - t\right) dt = \left[\frac{2}{3}t^3 - \frac{1}{2}t^2\right]_{-1}^{1} = \frac{4}{3}$$

and

$$\begin{aligned} \int_{C_2} f(x,y)dy &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\cos^2 t + \sin^2 t - \cos t\right)\cos t dt = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\cos t - \cos^2 t\right) dt \\ &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\cos t - \frac{1+\cos 2t}{2}\right) dt = \left[\sin t - \frac{1}{2}t - \frac{1}{4}\sin 2t\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = -\pi - \sqrt{2} \end{aligned}$$

Therefore $\int_C f(x,y)dy = \int_{C_1} f(x,y)dy + \int_{C_2} f(x,y)dy = \frac{4}{3} - \pi - \sqrt{2}$

The following definition summarizes what we introduced in Example 1.

Definition 7 *Let $f(x,y,z)$ be a given function and C be a curve in the domain of f defined by $\mathbf{c}(t) =< x(t), y(t), z(t) >$, $a \leq t \leq b$. Divide the interval $[a,b]$ into smaller subintervals $[t_0, t_1]$, $[t_1, t_2]$, $\ldots$, $[t_{n-1}, t_n]$ where $a = t_0 < t_1 < \cdots < t_n = b$. Form the sums*

$$\sum_{i=1}^{n} f(x(t_i), y(t_i), z(t_i))[x(t_{i+1}) - x(t_i)], \quad \sum_{i=1}^{n} f(x(t_i), y(t_i), z(t_i))[y(t_{i+1}) - y(t_i)], \quad \text{and}$$

$$\sum_{i=1}^{n} f(x(t_i), y(t_i), z(t_i))[z(t_{i+1}) - z(t_i)]$$

The limit of $\sum_{i=1}^{n} f(x(t_i), y(t_i), z(t_i))[x(t_{i+1}) - x(t_i)]$ *as the lengths of all the subintervals* $[t_i, t_{i+1}]$ *shrink to zero is called the line integral of f along C with respect to x and is denoted by* $\int_C f(x,y,z)dx$. *If $x(t)$ has a continuous derivative then an application of the Mean Value Theorem gives*

$$\int_C f(x,y,z)dx = \int_a^b f(x(t), y(t), z(t))x'(t)dt. \qquad (4)$$

The limit of $\sum_{i=1}^{n} f(x(t_i), y(t_i), z(t_i))[y(t_{i+1}) - y(t_i)]$ *as the lengths of all the subintervals* $[t_i, t_{i+1}]$ *shrink to zero is called the integral of f along C with respect to y and is denoted by* $\int_C f(x,y,z)dy$. *If $y(t)$ has a continuous derivative then an application of the Mean Value Theorem gives*

$$\int_C f(x,y,z)dy = \int_a^b f(x(t), y(t), z(t))y'(t)dt.$$

The limit of $\sum_{i=1}^{n} f(x(t_i), y(t_i), z(t_i))[z(t_{i+1}) - z(t_i)]$ *as the lengths of all the subintervals* $[t_i, t_{i+1}]$ *shrink to zero is called the integral of f along C with respect to z and is denoted by* $\int_C f(x,y,z)dz$. *If $z(t)$ has a continuous derivative then an application of the Mean Value Theorem gives*

$$\int_C f(x,y,z)zx = \int_a^b f(x(t), y(t), z(t))z'(t)dt.$$

Exercise 8

1. *Evaluate the given integral:*

   (a) $\int_C (xy + 3x - zy)\, dz$ *where C is the line segment from $(0,0,0)$ to $(4,3,2)$.*

   (b) $\int_C (xy - y^2)\, dx$ *where C is the line segment from $(-4,3)$ to $(1,5)$.*

   (c) $\int_C x^2 dy - 4xydx$ *where C is the semicircle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$.*

   (d) $\int_C f(x,y)dx$ *where $f(x,y) = x^2 + y^2 - x$ and C is the curve in Example 6 consisting of the line segment from $(-1,-1)$ to $(1,1)$ and the part of the circle $x^2 + y^2 = 2$ above the line $y = x$.*

2. *Let C_1 be the line segment joining $(0,0)$ to $(1,1)$, C_2 be the part of the curve $y = \sqrt{x}$ from $(1,1)$ to $(0,0)$ and $C = C_1 + C_2$. Show that* $\int_C 3xy^2 dx = -\frac{1}{4}$ *and* $\int_C 2x^2 dy = \frac{4}{15}$.

3. *In the example we gave to introduce line integrals with respect to given variables, we essentially showed that if* $C = \mathbf{c}(t) =< x(t), y(t), z(t) >$, $a \leq t \leq b$ *is a given curve and a body is acted on by a force* $\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$ *when it is at a point* (x,y,z) *in space then the work done to move the body from* $\mathbf{c}(a)$ *to a point* $\mathbf{c}(b)$ *along the curve is*

$$\int_C F_1(x,y,z)dx + F_2(x,y,z)dy + F_3(x,y,z)dz$$

Assume that the functions $x(t)$, $y(t)$, $z(t)$ *have continuous derivatives. Show that if* $\mathbf{F}(x,y,z)$ *is orthogonal to the tangent vector* $\mathbf{c}'(t)$ *at every point of the curve then the work done is zero.*

4. *Prove that if* $(x_1(t), y_1(t), z_1(t))$, $a \leq t \leq b$ *and* $(x_2(s), y_2(s), z_2(s))$, $c \leq s \leq d$ *are one-to-one parametrizations of* C *and* $f(x,y,z)$ *is defined on* C *then*

$$\int_a^b f\left(x_1(t), y_1(t), z_1(t)\right) x_1'(t)dt = \int_c^d f\left(x_2(s), y_2(s), z_2(s)\right) x_2'(s)ds$$

In other words, the line integral of f *along* C *with respect to* x *does not depend on the parametrization of* C.