Polynomials

So far we have considered the linear functions $f(x) = ax + b$ and the quadratic functions $g(x) = ax^2 + bx + c$ where a, b and c are constants. They are special cases of the polynomials, which have formulas of the form

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + rx + t$$

where a, b, c, ..., r, t are constants. Examples:

$$f(x) = 4x^3 - 7x^2 - 5x + 2 \qquad g(x) = 5x^4 + 6x^3 - 5x + 15 \qquad h(x) = x^4 - 3x - 12$$

The highest exponent n in the expression $ax^n + bx^{n-1} + cx^{n-2} + \cdots + rx + t$ is called the degree of the polynomial. Thus $f(x) = 4x^3 - 7x^2 - 5x + 2$ is a polynomial of degree 3 and $h(x) = x^4 - 3x - 12$ is a polynomial of degree 4.

Algebraic Operations on Polynomials

To add two or subtract two polynomials, add or subtract the like terms then simplify.

Example 1 *If* $f(x) = 5x^4 - 3x^2 + 2x - 3$, $g(x) = 6x^3 - 5x^2 - 3x$, $h(x) = x^4 - x^2 + 1$, $u(x) = 7x$ *then:*

1. *The sum of* f *and* g *is* $f + g$ *with formula* $(f + g)(x) = \left(5x^4 - 3x^2 + 2x - 3\right) + \left(6x^3 - 5x^2 - 3x\right)$ *which simplifies, on combining like terms, to* $5x^4 + 6x^3 - 8x^2 - x - 3$.
2. $g - h$ *has formula* $(g - h)(x) = -x^4 + 6x^3 - 4x^2 - 3x - 1$
3. $h + u$ *has formula* $(h + u)(x) = x^4 - x^2 + 7x + 1$
4. $u - f$ *has formula* $(u - f)(x) = -5x^4 + 3x^2 + 5x + 3$

To multiply polynomials, do the usual multiplication of algebraic expressions then combine like terms

Example 2 *If* $f(x) = x^2 - 5x + 3$, $g(x) = 5x^4 - 8x^3$ *and* $h(x) = x + 4$ *then*

1. *The product of* f *and* g *is written as* fg *and has formula*

$$(fg)(x) = \left(x^2 - 5x + 3\right)\left(5x^4 - 8x^3\right) = 5x^6 - 33x^5 + 65x^4 - 24x^3$$

2. *The product of* g *and* h *is* gh *with formula* $(gh)(x) = \left(5x^4 - 8x^3\right)(x + 4) = 5x^5 - 4x^4 - 32x^3$

Dividing polynomials is similar to dividing positive integers. You may recall that we generally divide a given positive integer n by a smaller positive integer d to get a quotient q and a remainder r that is smaller than the divisor d. For example, when we divide 149 by 11 we get a quotient 13 and remainder 6, then write

$$\frac{149}{11} = 13 + \frac{6}{11}$$

Alternatively, we may multiply both sides of the above identity by 11 to get

$$149 = 11 \times 13 + 6$$

In the case of polynomials, we divide a given polynomial $P(x)$ by another polynomial $D(x)$ **with a lower degree**, to get a quotient $Q(x)$ and a remainder $R(x)$ **that has a lower degree than that of** $D(x)$. Since division is repeated subtraction, we essentially keep on subtracting multiples of $D(x)$ until we get a remainder that has a lower degree than that of $D(x)$. We do the subtractions systematically as shown in the following example:

Example 3 *To divide* $P(x) = 3x^4 - 5x^3 + x + 1$ *by* $D(x) = x^2 + 1$*:*

As pointed out above, the strategy is to keep on subtracting multiples of x^2+1 until we get a remainder that has a lower degree than the degree of x^2+1, (which is 2). We should start by getting rid of the term $3x^4$ with the highest power. We have to multiply x^2+1 by $3x^2$ and subtract the result from $3x^4-5x^3+x+1$. We get:

$$3x^4-5x^3+x+1-3x^2\left(x^2+1\right)=-5x^3-3x^2+x+1$$

The degree of the remainder is still bigger than 2, (the degree of the divisor). The next step is to get rid of the term $-5x^3$. We multiply x^2+1 by $-5x$ and subtract the result from $-5x^3-3x^2+x+1$. We end up with

$$-5x^3-3x^2+x+1-\left[-5x\left(x^2+1\right)\right]=-3x^2+6x+1$$

Again the degree of the remainder is not smaller than the degree of $D(x)$, therefore we must continue subtracting. Clearly, we should multiply x^2+1 by -3 and subtract the result from $-3x^2+6x+1$. The result is

$$-3x^2+6x+1-\left[-3\left(x^2+1\right)\right]=6x+4$$

This time the degree of the remainder is lower than 2, therefore we stop. The conclusion is that when we divide $3x^4-5x^3+x+1$ by x^2+1 we get a quotient $3x^2-5x-3$ and a remainder $6x+4$. We may write this as

$$\frac{3x^4-5x^3+x+1}{x^2+1}=3x^2-5x-3+\frac{6x+4}{x^2+1}$$

OR, (if we multiply both sides by x^2+1),

$$3x^4-5x^3+x+1=\left(x^2+1\right)\left(3x^2-5x-3\right)+6x+4$$

In practice we divide systematically in a table, the way you did long division of whole numbers as shown below:

$$
\begin{array}{r|l}
 & 3x^2-5x-3 \\ \hline
x^2+1 & 3x^4-5x^3+x+1 \\
 & \underline{3x^4+3x^2} \\
 & \quad -5x^3-3x^2+x+1 \\
 & \quad \underline{-5x^3-5x} \\
 & \qquad -3x^2+6x+1 \\
 & \qquad \underline{-3x^2-3} \\
 & \qquad\qquad 6x+4
\end{array}
$$

Exercise 4 *In each case below, you are given polynomials $P(x)$ and $D(x)$. Divide $P(x)$ by $D(x)$ and write your answer in the form $P(x)=D(x)Q(x)+R$ where $Q(x)$ is the quotient and $R(x)$ is the remainder.*

1. $P(x)=6x^3+3x^2-4x+2,\ \ D(x)=x^2+1$
2. $P(x)=x^4-x^2+x-3,\ \ D(x)=x^2-2$
3. $P(x)=2x^3+5x^2+3x+3,\ \ D(x)=x-3$

4. $P(x) = x^4 + 7x^3 + 12x^2 + x + 4, \;\; D(x) = x + 3$

5. $P(x) = x^4 + x^3 - 5x^2 + x - 6, \;\; D(x) = x - 3$

6. $P(x) = 3x^3 - 2x^2 + 5x + 8, \;\; D(x) = x - 1$

7. $P(x) = x^3 + 125, \;\; D(x) = x + 5$

The Remainder Theorem

When we divide a polynomial $P(x)$ by a linear polynomial of the special form $D(x) = (x - c)$ where c is a constant, we get a quotient $Q(x)$ and a remainder that must be a constant r, (because it must have degree zero). We may write the result as

$$P(x) = (x - c)\, Q(x) + r \tag{1}$$

Notice that when we substitute $x = c$ in (1) we get

$$P(c) = r$$

In other words, the remainder when a polynomial $P(x)$ is divided by a linear polynomial of the special form $D(x) = (x - c)$, is the value $P(c)$, of P when x equals c. This result is called the REMAINDER THEOREM.

Example 5 *The remainder when* $P(x) = 4x^3 + 7x^2 - 5x + 2$ *is divided by* $(x - 2)$ *is*

$$P(2) = 4\,(8) + 7\,(4) - 5\,(2) + 2 = 52$$

Example 6 *The remainder when* $P(x) = x^5 + 6x^3 - 7x - 5$ *is divided by* $(x - 1)$ *is*

$$P(1) = 1 + 6 - 7 - 5 = -5$$

Example 7 *The remainder when* $P(x) = x^{53} + 6x^{34} - 7x - 5$ *is divided by* $(x + 1)$ *is*

$$P(-1) = -1 + 6 - 7\,(-1) - 5 = 7$$

Note that we have to write $(x + 1)$ *as* $(x - (-1))$ *in order to get the number that has to be substituted into the polynomial.*

Example 8 *The remainder when* $P(x) = x^3 - 2x + 4$ *is divided by* $(x + 2)$ *is*

$$P(-2) = (-8) - 2\,(-2) + 4 = 0$$

This means that $(x+2)$ *divides* $x^3 - 2x + 4$ *because there is no remainder. In other words,* $(x+2)$ *is a factor of* $x^3 - 2x + 4$.

In general, if $P(x)$ is a given polynomial and $P(c) = 0$ then $(x - c)$ is a factor of $P(x)$. This gives us a method of factoring some polynomials. Here is an example:

Example 9 *To factor* $P(x) = 2x^3 + 3x^2 - 3x - 2$:

> *We look for numbers* c *such that* $P(c) = 0$. *We should look among the factors of* -2 *for the simple reason that if* $(x - c)$ *divides* $2x^3 + 3x^2 - 3x - 2$ *then there is a polynomial* $px^2 + qx + r$ *such that*
>
> $$2x^3 + 3x^2 - 3x - 2 = (x - c)\left(px^2 + qx + r\right)$$
>
> *and when we expand the right hand side we get*
>
> $$2x^3 + 3x^2 - 3x - 2 = px^3 + \cdots - cr$$
>
> *Since the constant terms of the two polynomials must be the same,* $-2 = -cr$, *therefore* c *divides* -2.

The factors of -2 *are* 1, -1, 2 *and* -2. *By trial - and - error, we find that* $P(1) = 0$ *therefore* $(x-1)$ *must be a factor of* $P(x) = 2x^3 + 3x^2 - 3x - 2$. *When we divide* $2x^3 + 3x^2 - 3x - 2$ *by* $(x-1)$ *we get a quotient* $2x^2 + 5x + 2$ *and, as expected, no remainder. Therefore*

$$2x^3 + 3x^2 - 3x - 2 = (x-1)\left(2x^2 + 5x + 2\right)$$

We now factor the quadratic term $2x^2 + 5x + 2$ *in the usual way to get* $2x^2 + 5x + 2 = (2x+1)(x+2)$. *Therefore*

$$2x^3 + 3x^2 - 3x - 2 = (x-1)(2x+1)(x+2)$$

Exercise 10

1. *Use the remainder theorem to find the remainder when:*
   (a) $P(x) = x^4 - 4x^3 + 5x^2 - 6x + 1$ *is divided by* $D(x) = (x+2)$.
   (b) $P(x) = x^{41} - 3x^{24} + 5x - 9$ *is divided by* $D(x) = (x+1)$
   (c) $P(x) = x^{25} - 5x^{19} - 3x^4 + 7$ *is divided by* $D(x) = (x-1)$

2. *Use the Remainder Theorem to find a zero of the given polynomial, then divide the polynomial by an appropriate factor, and finally factor the polynomial.*
   (a) $P(x) = 2x^3 + 3x^2 - 11x - 6$
   (b) $P(x) = 3x^3 - 13x^2 + 13x - 3$
   (c) $P(x) = x^3 + 7x^2 + 14x + 8$
   (d) $P(x) = x^3 + 2x^2 - 5x - 6$
   (e) $P(x) = 6x^3 + 5x^2 - 12x + 4$