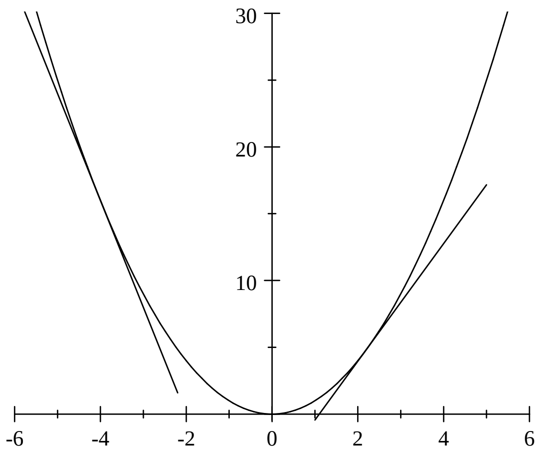
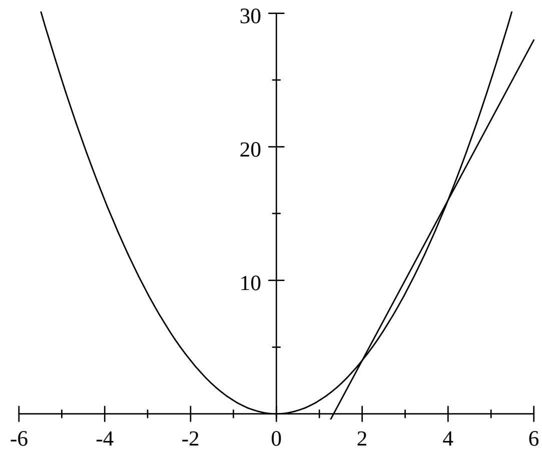
A problem we can solve if we know how to calculate slopes of tangents to curves

The first part of this course is devoted to calculating slopes of tangents, and using them to solve a variety of problems. You should have an intuitive idea of a tangent to the graph of a given function f. It is a line that "lies flat on the graph". In the figures below two tangents and a line that is not a tangent to the graph of $f(x) = x^2$ are shown.



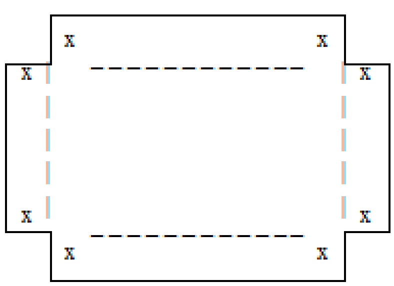
Two tangents to the graph

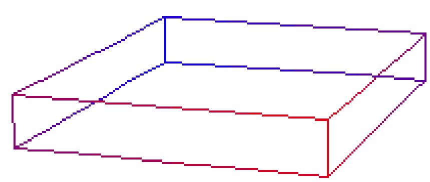


Not a tangent

There are many problems one can solve if one knows how to calculate slopes of tangents to graphs. Here is an example that requires you to maximize a volume:

You are the production manager of a small metal factory. You receive a shipment of 20 cm by 26 cm metal plates to be transformed into boxes without tops, by removing identical squares from each corner of a plate then fold along the dotted lines to get a box with no cover as shown below.





Box with no cover

The owner of the shipment wants you to produce boxes with the largest possible volume. What size of squares will you recommend to be cut from each plate?

The most natural step to take is to construct such paper boxes for different values of x. A creative student suggested that, instead of cutting out four squares of length x, it suffices to cut four slits of length x positioned as shown in Figure (a), fold along the dotted lines shown in Figure (b) then use a stapler or glue to stick the box. Cut four slits of length $x = 5$ units each in the given 20 by 26 rectangle, make a box and calculate its volume.

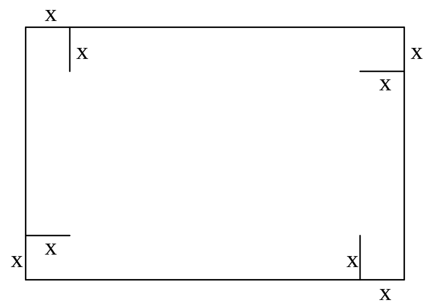


Figure (a)

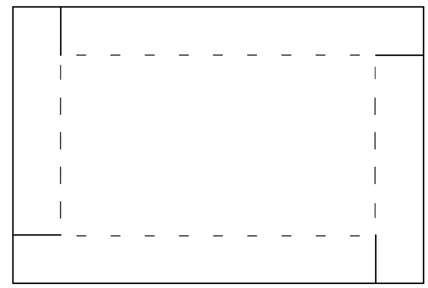


Figure (b)

Make another box by cutting slits of length $x = 6$ units and also calculate its volume.

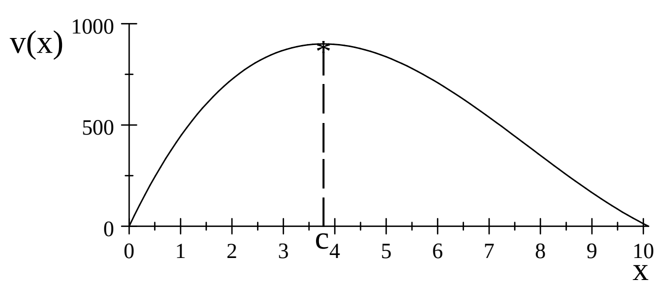
Of course the volume of a box you make depends on the size of the square you remove. Most probably you have figured out a method of calculating the volume of a box formed when a particular size of square is cut out without first constructing the box. Use it to complete the following table:

Length of the square removed	1	2	3	4	5	5.5	6	6.5	7
Length of the box formed		22							
Width of the box formed		16							
Volume of the box formed		704							

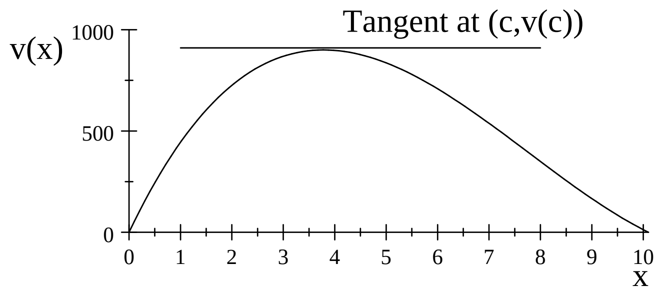
The length x of a square that may be removed from each corner must be less than 10 cm, (explain why), and it must be positive. This places x in the interval $[0, 10]$. Therefore you must recommend a length c, between 0 and 10, of a square that should be cut from each corner to form the box with the largest possible volume. It is impossible to try out all the numbers between 0 and 10 because they are infinitely many, therefore a different approach is necessary. Plotting a graph of the function that gives the volume of the box formed in terms of the length x of each square removed is one option. To obtain its formula, note that if you remove a square of length x from each of the 4 corners then the box you form has length $(20 - 2x)$ cm, width $(26 - 2x)$ cm, and height x cm., therefore its volume is

$$v(x) = x\,(20 - 2x)\,(26 - 2x) = 4x\,(10 - x)\,(13 - x)\,, \quad 0 \le x \le 10$$

We may remove parentheses to get $v(x) = 4x^3 - 92x^2 + 520x$. Its graph is shown below. It does not provide the exact answer, but it suggests that c is between 3.5 cm and 4 cm. It also provides the following insight into what is going on: If you do not cut off anything, (which corresponds to $x = 0$), you get a zero volume. As you increase the size of the square you cut off, the volume of the box you get increases, but not for ever. When x gets to the required value c, (which we do not know at the moment), you get the maximum possible volume. Then further increases in x give smaller volumes, eventually shrinking to 0 when $x = 10$ cm.



A property that distinguishes the number c we are looking for from all the other numbers between 0 and 10 is that the tangent to the graph of v at the point $(c, v(c))$ is horizontal, (see the figure below), hence its slope is 0.



This suggests that to solve this problem, it is sufficient to devise a method of determining the slopes of tangents to graphs of functions. With such a tool at our disposal, the problem is solved by determining the number c between 0 and 10 such that the slope of the tangent to the graph of v at $(c, v(c))$ is zero. That tool has already been developed and refined. It is called **differential calculus**, and it is the main focus of the next couple of lectures. We will soon learn, (using this tool), that the slope of the tangent to the graph of $v(x) = 4x^3 - 92x^2 + 520x$ at any point $(x, v(x))$ is $12x^2 - 184x + 520$. Therefore, the required number c is the solution of the equation

$$12x^2 - 184x + 520 = 0$$

which is between 0 and 10. This equation may be solved by using the quadratic formula, and the result is

$$x = \frac{184 \pm \sqrt{184^2 - 4(12)(520)}}{24}$$

Of the two solutions, $x = 3.74$ and $x = 11.64$, (rounded off to 2 decimal places), it is the first one which is between 0 and 10. It follows that the largest possible volume is approximately

$$4\,(3.74)^3 - 92\,(3.74)^2 + 520\,(3.74) = 867.20 \text{ cubic centimeters, (to 2 decimal places).}$$

For further practice, solve the same problem, but this time, assume that you receive a shipment of 17 cm by 24 cm plates. Start by completing the table below

Length of the square removed	1	2	3	4	5.5	5.5	6.5	x
Length of the box formed								
Width of the box formed								
Volume of the box formed								

You may use the fact, (to be derived soon), that the slope of the tangent to the graph of the function $v(x) = 4x^3 - 82x^2 + 408x$ at a point $(x, v(x))$ is $12x^2 - 164x + 408$.