Linear Functions

The following is an example that leads directly to a linear function

Example 1 *You wish to start a small profitable business selling pizzas. You have to do some cost analysis. Say you have to pay the following:*

Fixed monthly costs consisting of:

Rent: \$700.00 *Water: \$40.00* *Manager: \$3000.00* *Insurance: \$68.00*

Variable costs consisting of:

Ingredients *Electricity* *Hourly labor*

They are lumped together into one cost per pizza which is \$4.00. In other words, the ingredients, electricity and hourly labor costs contribute 4 dollars to the cost of each pizza.

Due to space and labor limitations, you cannot make more than 1200 pizzas a month. Complete the following table showing how the monthly expenditure is related to the number of pizzas produced per month:

# of pizzas you produce a month	100	230	304	500	560	755	1200
Your monthly expenditure		4728					

Denote it by E , (an abbreviation for expenditure). Show that it has formula

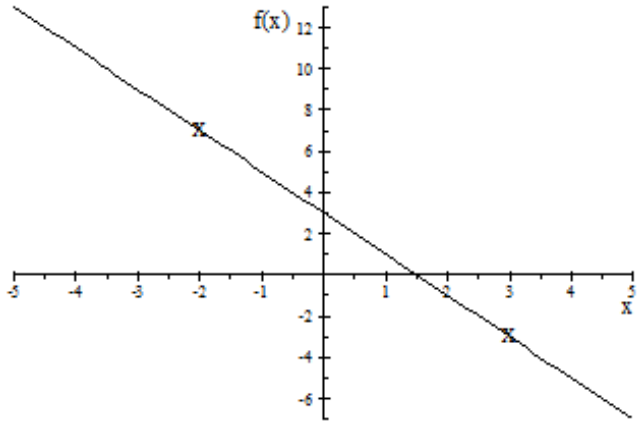
$$E(x) = 4x + 3808$$

This is an example of a linear function.

In general, a linear function is any function with a formula of the form $f(x) = ax + b$ where a and b are constants. Its graph is a straight line and it has the distinct property that whenever the independent variable increases by one unit, the dependent variable changes by a units. For this reason, a is called the slope of the line. (In general, when the independent variable changes by h units then the dependent variable changes by ah units.) Because the graph crosses the vertical axis, (i.e. it crosses the y-axis), at $(0, b)$, b is called its y-intercept. (In the case of $E(x) = 4x + 3808$ derived above, if you increase the number of pizzas produced monthly by 1 then the monthly expenditure increases by 4 dollars, therefore slope of its graph is 4. The y-intercept is $b = 3808$.)

Drawing the graph of a linear function f is easy; all you need are (i) two numbers u and v in its domain and (ii) the points $(u, f(u))$ and $(v, f(v))$. The graph is the straight line joining them.

Example 2 *To plot the graph of the linear function $f(x) = -2x + 3$: The domain of this function is the set of all the real numbers. Pick two convenient ones e.g. -2 and 3. Clearly $f(-2) = 7$ and $f(3) = -3$. Therefore the graph of f is the straight line joining $(-2, 7)$ and $(3, -3)$.*



Direct Proportionality

We say that one quantity is directly proportional to another quantity if:

When one of them doubles, the other one also doubles,

When one of them is halved, the other one is also halved,

When one triples, the other one triples,

In general, when one of them changes by a certain factor, so does the other.

Example 3 *The amount of money you pay at a gas pump for gas is directly proportional to the number of gallons you buy. Say the price per gallon, to the nearest cent, is \$3.12. If you buy 3 gallons, the price tag will be \$9.36. If you triple the purchase to 9 gallons, the price tag also triples to \$28.08. The table below shows how the two quantities are related*

Number of gallons you buy	0	1	2	5.5	11	x
The price you pay, in dollars	0	3.12	6.24	17.16	34.32	$3.12x$

This table of sample values is a function. Denote it by P, (for price). Then

$$P(x) = 3.12x$$

We now generalize: The value of one quantity is directly proportional to the value of another quantity if, (like the price and number of gallons in Example (3) above), one value is a constant multiple of the other value. Here are two more examples:

Example 4 *Consider a train travelling at a constant speed of 76 miles per hour. As the table below shows, the distance it has travelled is directly proportional to the number of hours it has been travelling for.*

Number of hours since it started travelling	0	1	2	3	4.8	x
The distance it has travelled in miles	0	76	152	228	364.8	$76x$

If we denote the number of miles it has travelled in x hours by $D(x)$ then

$$D(x) = 76x$$

Example 5 *Consider a part-time employee who is paid \$18.00 for every hour he works. The amount of money he earned in a month is directly proportional to the number of hours he worked during the month.*

Number of hours he worked during the month	0	10	20	60	120	x
Amount of money, in dollars, he earned	0	180	360	1080	2160	$18x$

If we denote the amount of money he earned for working x hours by $E(x)$ then

$$E(x) = 18x$$

To summarize, a quantity V is directly proportional to a quantity x if V is a constant multiple of x. Thus there is a number k such that

$$V = kx$$

The number k is called the constant of proportionality. In Example (3), the constant of proportionality is 3.12. In Examples (4) and (5), the constants of proportionality are 76 and 18 respectively. The graph of V is a straight line through the origin $(0, 0)$ and it has slope k.

Example 6 *Jane bought* 4.3 *pounds of beef for* \$12.47. *How much money would she have paid for* 3.4 *pounds of the same beef?*

> ***Solution****: The amount of money one pays for the beef is directly proportional to the number of pounds one buys. One way to solve the problem is to find the constant of proportionality, which is the cost of 1 pound of the beef. Since* 4.3 *pounds cost* 12.47 *dollars, 1 pound costs* $\frac{12.47}{4.3}$ *dollars. This is the constant of proportionality. It follows that* 3.4 *pounds cost*
>
> $$\frac{12.47}{4.3} \times 3.4 \quad \textit{dollars}$$
>
> *This simplifies to* \$9.86.

Exercise 7

1. *What is the domain of the function in Example 1?*
2. *You are given the linear functions* $f(x) = 3x + 2$ *and* $g(x) = -x + 6$.
   (a) *For each function, determine the slope and y-intercept of its graph.*
   (b) *Determine the point of intersection of their graphs.*
3. *Given the linear function* $f(x) = 2x + 7$, *do the following:*
   (a) *Determine* $f(3)$, $f(-\frac{2}{5})$ *and* $f(\frac{a}{b})$ *where* $b \neq 0$.
   (b) *Determine a formula for the inverse of* f.
4. *Given the linear functions* $f(x) = 3x - 5$ *and* $g(x) = -\frac{2}{3}x$, *give a formula for* $f \circ g(x)$ *and* $g \circ f(x)$.
5. *Given the linear functions* $f(x) = 2x - 5$, $h(x) = \frac{3}{4}x$ *and* $g(x) = -\frac{5}{4}x$, *give a formula for* $f \circ g \circ h(x)$ *and a formula for* $h \circ g \circ f(x)$.
6. *Determine the equation of the linear function whose graph has the stated properties:*
   (a) *Its slope is* 3 *and its y-intercept is* -4.
   (b) *Its slope is* -4 *and its y-intercept is 3.*
   (c) *It passes through* $(-2, 3)$ *and* $(4, -5)$.
   (d) *It has slope* $-\frac{4}{3}$ *and passes through* $(1, 1)$.
   (e) *Its y-intercept is 2 and it passes through* $\left(\frac{1}{3}, \frac{5}{4}\right)$.
7. *If three pizzas cost* \$7.40, *how much do 8 pizzas of the same size cost?*
8. *Three pens cost \$2.20. I have* \$16.00 *in cash. How many pens can I buy with that* \$16.00 *and what will be my balance?*