Changing Variables in a Double Integral

Changing variables, (i.e. integrating by substitution), in a definite integral

$$\int_a^b f(x)dx$$

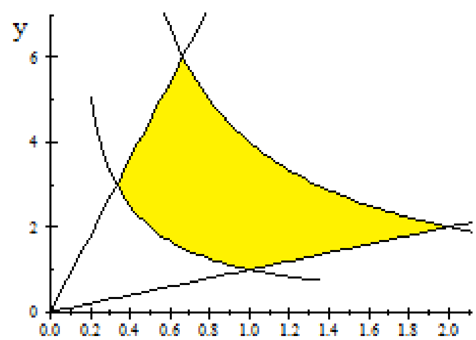
of a function f of one variable boiled down to determining a suitable function g and an interval $[c,d]$ that g maps onto $[a,b]$ then integrate $\int_c^d f \circ g(u) g'(u) du$ instead of $\int_a^b f(x)dx$. We called $g'(u)$ a scaling factor for the change of variable. One had to choose g such that $f \circ g(u)g'(u)$ has an antiderivative that could be determined by inspection.

We follow similar steps to change variables in a function of two variables. More precisely, given an integral

$$\iint_R f(x,y)dA,$$

(most probably the type that eludes our attempts to evaluate by iteration), we look for a suitable function g and a set W that g maps onto R. Then, instead of integrating f over R, we integrate $(f \circ g)$ $\times$(a *scaling factor*) over the W. We will soon give an expression for the *scaling factor* in terms of g, but before doing that, here is an example:

Example 1 *Let R be the region in the first quadrant enclosed by the curves $xy = 1$, $xy = 4$ and the two lines $y = x$ and $y = 9x$. The "corners" of R have coordinates $\left(\frac{1}{3}, 3\right)$, $(1,1)$, $(2,2)$ and $\left(\frac{2}{3}, 6\right)$.*



The set R

Let $f(x,y) = \sqrt{xy+4}$. Say we decide to evaluate $\iint_R f(x,y)dA$ by iteration. If we fix x we get a function of one variable y but, this time, its domain depends more elaborately on where x is in the interval $\left[\frac{1}{3}, 2\right]$. If $\frac{1}{3} \le x \le \frac{2}{3}$, the domain is $\left[\frac{1}{x}, 9x\right]$. If $\frac{2}{3} \le x \le 1$, the domain is $\left[\frac{1}{x}, \frac{4}{x}\right]$, and if $1 \le x \le 2$, the domain is $\left[x, \frac{4}{x}\right]$. Therefore we have to evaluate three separate integrals which are

$$\int_{\frac{1}{3}}^{\frac{2}{3}} \int_{\frac{1}{x}}^{9x} \left(\sqrt{xy+4}\right) dydx, \quad \int_{\frac{2}{3}}^{1} \int_{\frac{1}{x}}^{\frac{4}{x}} \left(\sqrt{xy+4}\right) dydx, \quad \text{and} \quad \int_{1}^{2} \int_{x}^{\frac{4}{x}} \left(\sqrt{xy+4}\right) dydx$$

It turns out that none of these three integrals is a "walk-over". For example,

$$\int_{\frac{1}{3}}^{\frac{2}{3}} \int_{\frac{1}{x}}^{9x} \left(\sqrt{xy+4}\right) dydx = \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\frac{2}{3} \left[\frac{1}{x} (xy+4)^{3/2} \right]_{\frac{1}{x}}^{9x} \right) dx = \frac{2}{3} \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\frac{\left(9x^2+4\right)^{3/2}}{x} - \frac{5^{3/2}}{x} \right) dx$$

Faced with such challenging integrals, it pays to look for a mapping g that maps a relatively simple set W onto R and is such that $f \circ g$ has a simpler formula. Then, instead of integrating f over R, we integrates

$$f \circ g \times (a\ scaling\ factor)$$

over W. We show ahead that the mapping $g(u,v) = \left(u^{\frac{1}{2}}v^{-\frac{1}{2}}, u^{\frac{1}{2}}v^{\frac{1}{2}}\right)$, (we explain how to get it), maps the rectangle $\{(u,v) : 1 \leq u \leq 4$ and $1 \leq v \leq 9\}$ onto R. It is easy to verify that $f \circ g(u,v) = \sqrt{u+4}$. Therefore, instead of integrating $f(x,y)$ over R, we integrate the simpler function

$$f \circ g \times (a\ scaling\ factor) = \left(\sqrt{u+4}\right) \times (a\ scaling\ factor)$$

over W. This turns out to be a much easier task.

The scaling Factor

Theorem 2 *Let R and W be subsets of $\mathbb{R}^2$, $g(u,v) = (g_1(u,v), g_2(u,v))$ be a one-to-one function that maps W onto R and has continuous partial derivatives $\dfrac{\partial g_1}{\partial u}$, $\dfrac{\partial g_2}{\partial u}$, $\dfrac{\partial g_1}{\partial v}$ and $\dfrac{\partial g_2}{\partial v}$. Consider the the determinant*

$$\begin{vmatrix} \dfrac{\partial g_1(u,v)}{\partial u} & \dfrac{\partial g_1(u,v)}{\partial v} \\ \dfrac{\partial g_2(u,v)}{\partial u} & \dfrac{\partial g_2(u,v)}{\partial v} \end{vmatrix}$$

where all the partial derivatives are evaluated at the same point (u,v) in the domain of g. It is generally denoted by $\dfrac{\partial(g_1,g_2)}{\partial(u,v)}$ and called the Jacobian of the transformation $g(u,v)$. If a function $f(x,y)$ is integrable on R then

$$\iint\limits_R f(x,y)dA = \iint\limits_W f \circ g(u,v) \left|\frac{\partial(g_1,g_2)}{\partial(u,v)}\right| dA$$

Thus the scaling factor for the change of variables is $\left|\dfrac{\partial(g_1,g_2)}{\partial(u,v)}\right|$.

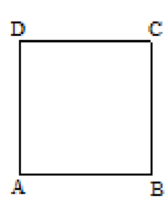
A rigorous proof of this statement requires heavy-duty tools which are not accessible till after a course in real analysis. The following is essentially an intuitive argument that tries to justify the assertion of the theorem. To simplify the argument, assume that W is a rectangle $\{(u,v) : a \leq u \leq b$ and $c \leq v \leq d\}$. Divide $[a,b]$ into n smaller subintervals $[u_0,u_1]$, $[u_1,u_2]$, ..., $[u_{n-1},u_n]$ of equal length $\triangle u = (b-a)/n$ where

$$a = u_0 < u_1 < u_2 < \cdots < u_{n-1} < u_n = b.$$

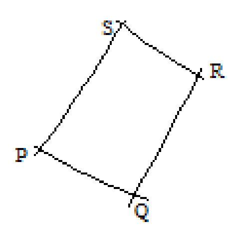
Also divide $[c,d]$ into m smaller subintervals $[v_0,v_1]$, $[v_1,v_2]$, ..., $[v_{m-1},v_m]$ of equal length $\triangle v = (d-c)/m$ where

$$c = v_0 < v_1 < v_2 < \cdots < v_{m-1} < v_m = d.$$

The mn rectangles $W_{ij} = \{(u,v) : u_i \leq u \leq u_{i+1}$ and $v_j \leq v \leq v_{j+1}\}$ partition divide W into smaller rectangles. Note that W_{ij} has area $\triangle W_{ij} = \triangle u \triangle v$. Since g is one-to-one and onto, their images $g(W_{ij})$ divide W into smaller regions, (which need not be rectangles). A typical rectangle W_{ij} with vertices at $A(u_i,v_j)$, $B(u_{i+1},v_j)$, $C(u_{i+1},v_{j+1})$ and $D(u_i,v_{j+1})$ is mapped onto a region $g(W_{ij})$ with "corners" at $P(g_1(u_i,v_j), g_2(u_i,v_j))$, $Q(g_1(u_{i+1},v_j), g_2(u_{i+1},v_j))$, $R(g_1(u_{i+1},v_{j+1}), g_2(u_{i+1},v_{j+1}))$ and $S(g_1(u_i,v_{j+1}), g_2(u_i,v_{j+1}))$.



Typical rectangle W_{ij}



Its image $g(W_{ij})$

The integral of f over R is the limit of the sums

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i, y_j\right) \times (Area\ of\ g(W_{ij}))$$

where (x_i, y_j) is a point in $g(W_{ij})$. An expression for the area of $g(W_{ij})$ may be hard to find, therefore we approximate it with a region whose area is more obvious. To do so, we appeal to the Mean Value Theorem for a function of one variable. Applying it to g_1 and the two points (u_{i+1}, v_j), (u_i, v_j) we conclude that there is a point (a_i, b_j) on the line segment joining (u_i, v_j) and (u_{i+1}, v_j) such that

$$g_1(u_{i+1}, v_j) - g_1(u_i, v_j) = (u_{i+1} - u_i)\frac{\partial g_1\left(a_i, b_j\right)}{\partial u} = \triangle u \frac{\partial g_1\left(a_i, b_j\right)}{\partial u}$$

Since $\dfrac{\partial g_1}{\partial u}$ is continuous, we approximate $\dfrac{\partial g_1\left(a_i, b_j\right)}{\partial u}$ with $\dfrac{\partial g_1\left(u_i, v_j\right)}{\partial u}$. Therefore

$$g_1(u_{i+1}, v_j) \simeq g_1(u_i, v_j) + \triangle u \frac{\partial g_1\left(u_i, v_j\right)}{\partial u}$$

We show in a similar way that

$$g_2(u_{i+1}, v_j) \simeq g_2(u_i, v_j) + \triangle u \frac{\partial g_2\left(u_i, v_j\right)}{\partial u}$$

$$g_1(u_i, v_{j+1}) \simeq g_1(u_i, v_j) + \triangle v \frac{\partial g_1\left(u_i, v_j\right)}{\partial u}$$

$$g_2(u_i, v_{j+1}) \simeq g_1(u_i, v_j) + \triangle v \frac{\partial g_1\left(u_i, vj\right)}{\partial u}$$

We now approximate $g(W_{ij})$ with the parallelogram that has adjacent sides PQ' and PS' where

P is the original point with coordinates $(g_1(u_i, v_j), g_2(u_i, v_j))$

Q' is a point close to Q that has coordinates $\left(g_1(u_i, v_j) + \triangle u \dfrac{\partial g_1\left(u_i, v_j\right)}{\partial u}, g_2(u_i, v_j) + \triangle u \dfrac{\partial g_2\left(u_i, v_j\right)}{\partial u}\right)$

S' is a point close to S that has coordinates $\left(g_1(u_i, v_j) + \triangle v \dfrac{\partial g_1\left(u_i, v_j\right)}{\partial v}, g_2(u_i, v_j) + \triangle v \dfrac{\partial g_2\left(u_i, v_j\right)}{\partial v}\right)$

Since $\vec{PQ'} =< \triangle u \dfrac{\partial g_1}{\partial u}, \triangle u \dfrac{\partial g_2}{\partial u} >$ and $\vec{PS'} =< \triangle v \dfrac{\partial g_1}{\partial v}, \triangle v \dfrac{\partial g_2}{\partial v} >$, the area of the parallelogram is the norm of the vector

$$\vec{PQ'} \times \vec{PS'} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \triangle u \dfrac{\partial g_1}{\partial u} & \triangle u \dfrac{\partial g_2}{\partial u} & 0 \\ \triangle v \dfrac{\partial g_1}{\partial v} & \triangle v \dfrac{\partial g_2}{\partial v} & 0 \end{vmatrix}$$

The norm is $\left|\dfrac{\partial(g_1, g_2)}{\partial(u, v)}\right| \triangle u \triangle v$. As expected, we denote $\triangle u \triangle v$ by $\triangle A_{ij}$. We have to pick a point (θ_i, α_j) from each region $g(W_{ij})$. The choice $(\theta_i, \alpha_j) = g(u_i, v_j)$ suffices. Therefore the integral is the limit of the sums

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f \circ g(u_i, v_j)\left|\frac{\partial(g_1, g_2)}{\partial(u, v)}\right| \triangle u \triangle v = \sum_{i=1}^{n}\sum_{j=1}^{m} f \circ g(u_i, v_j)\left|\frac{\partial(g_1, g_2)}{\partial(u, v)}\right| \triangle A_{ij}$$

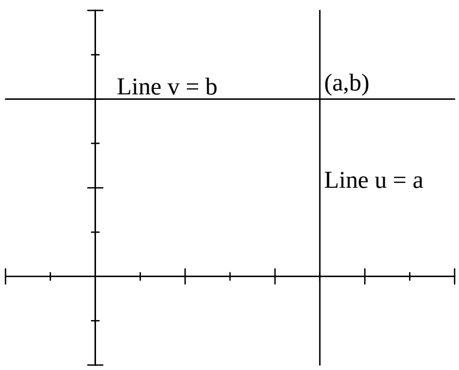
In other words,

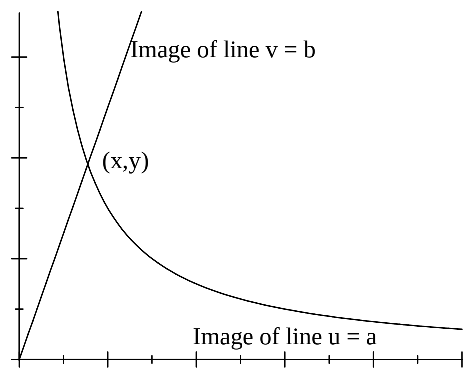
$$\iint_R f(x, y) dR = \lim_{\triangle u, \triangle v \to 0} \sum_{i=1}^{n}\sum_{j=1}^{m} f \circ g(u_i, v_j)\left|\frac{\partial(g_1, g_2)}{\partial(u, v)}\right| \triangle u \triangle v = \iint_W f \circ g(u, v)\left|\frac{\partial(g_1, g_2)}{\partial(u, v)}\right| dA.$$

Example 3 *To evaluate* $\iint_R \left(\sqrt{xy+4}\right) dA$ *in Example 1, we look for a mapping* $g(u,v)$ *that maps a rectangle with sides parallel to the coordinate axes onto the given region* R*. To this end fix a point* (a,b) *in the* u *-* v *plane. We look for a mapping that maps the vertical line line* $u=a$ *onto the curve* $y=\frac{a}{x}$ *and maps the horizontal line* $v=b$ *onto the line* $y=bx$*. Thus a point* (a,v) *on the vertical line is mapped onto the point with coordinates of the form* $\left(x,\frac{a}{x}\right)$*, (because it must be on the curve* $y=\frac{a}{x}$*). Similarly, a point* (u,b) *on the horizontal line is mapped onto a point of the form* (x,bx)*. It follows that the point* (a,b) *where the horizontal line intersects the vertical line is mapped into a point that satisfies the condition* $(x,bx)=\left(x,\frac{a}{x}\right)$ *which implies that*

$$bx=\frac{a}{x}$$

Solving gives $x=\left(\frac{a}{b}\right)^{\frac{1}{2}}=a^{\frac{1}{2}}b^{-\frac{1}{2}}$ *and* $bx=(ab)^{\frac{1}{2}}$*. Thus the mapping* g *we are looking for maps a point* (a,b) *onto the point* $\left(a^{\frac{1}{2}}b^{-\frac{1}{2}},(ab)^{\frac{1}{2}}\right)$*.*

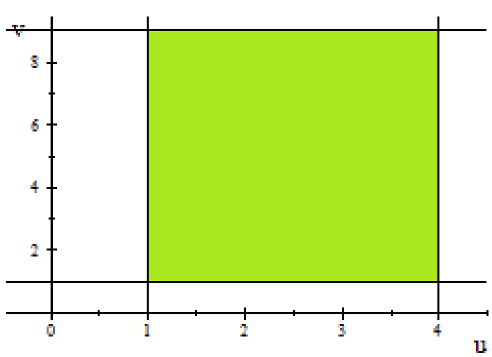




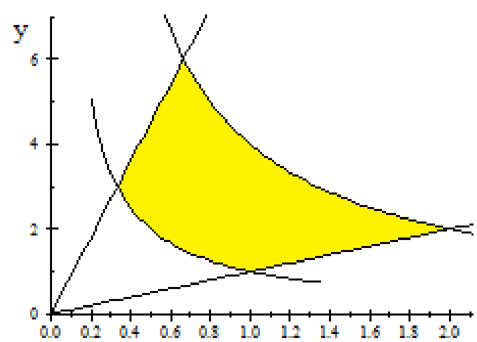
In general, the mapping sends (u,v) *onto* $\left(u^{\frac{1}{2}}v^{-\frac{1}{2}},u^{\frac{1}{2}}v^{\frac{1}{2}}\right)=(x,y)$*, therefore*

$$g(u,v)=\left(u^{\frac{1}{2}}v^{-\frac{1}{2}},u^{\frac{1}{2}}v^{\frac{1}{2}}\right)$$

It maps the vertical line $u=1$ *onto the curve* $y=\frac{1}{x}$ *or simply the curve* $xy=1$*. It maps the vertical line* $u=4$ *onto the curve* $xy=4$*. The horizontal line* $v=1$ *goes into the line* $y=x$ *and the horizontal line* $y=9$ *goes into the line* $y=9x$*. Now explain why the points inside the rectangle* $W=\{(u,v):1\le u\le 4 \text{ and } 1\le v\le 9\}$ *are mapped onto points inside* R*.*



The rectangle W



The set R

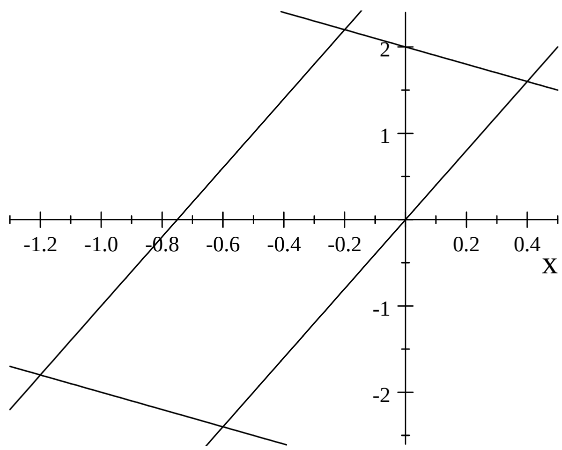
The Jacobian matrix for the transformation g is

$$\left|\frac{\partial(g_1,g_2)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} & -\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}} \\ \\ \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} \end{vmatrix} = \frac{1}{4v}$$

Therefore

$$\begin{aligned}\iint\limits_R \left(\sqrt{xy+4}\right) dA &= \iint\limits_W \frac{1}{4v} f\circ g(u,v) dA = \frac{1}{4}\iint\limits_W \frac{\sqrt{u+4}}{v} dA = \frac{1}{4}\int_1^4\int_1^9 \frac{\sqrt{u+4}}{v} dv du \\ &= \frac{1}{4}\int_1^4 \left(\sqrt{u+4}\right)[\ln v]_1^9 \, du = \frac{\ln 9}{4}\int_1^4 \left(\sqrt{u+4}\right) du = \frac{3\ln 9}{8}\left[(u+4)^{\frac{3}{2}}\right]_1^4 \\ & \frac{3\ln 9}{8}\left(8^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)\end{aligned}$$

Example 4 *Let R be the region, shown below, enclosed by the lines* $y = 4x$, $y = 4x+3$, $y = -x+2$ *and* $y = -x-3$. *Its vertices have coordinates* $(-1.2,-1.8)$, $(-0.6,-2.4)$, $(0.4,1.6)$ *and* $(-0.2,2.2)$.



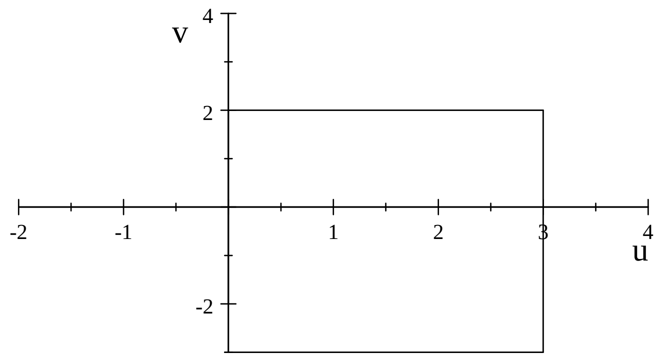
Let $f(x,y) = xy + x + y$. *We wish to determine* $\iint\limits_R f(x,y)dA$. *We face a similar problem to that in Example 3; we have to evaluate 3 different integrals. To avoid this, we introduce a function g that maps a rectangle with sides that are parallel to the coordinate axes R. An example is a function g that maps a vertical line* $u = a$ *onto the line* $y = 4x + a$ *and maps a horizontal line* $v = b$ *onto the line* $y = -x + b$. *Thus points* (a, v) *are mapped onto points of the form* $(x, 4x + a)$ *and points* (u, b) *are mapped onto points of the form* $(x, -x + b)$. *In particular, the point* (a, b) *is mapped onto a point that satisfies the condition* $(x, 4x + a) = (x, -x + b)$. *Therefore*

$$4x + a = -x + b$$

Solving gives $x = \frac{1}{5}(b-a)$ *and* $-x+b = \frac{1}{5}(4b+a)$. *Thus the point* (a, b) *is mapped onto* $\left(\frac{1}{5}(b-a), \frac{1}{5}(4b+a)\right)$. *In general, a point* (u, v) *is mapped onto onto* $\left(\frac{1}{5}(v-u), \frac{1}{5}(4v+u)\right) = (x, y)$, *therefore the map g we want is defined by*

$$g(u,v) = \left(\tfrac{1}{5}(v-u), \tfrac{1}{5}(4v+u)\right)$$

It maps the rectangle W, (shown below), with vertices at $(0,-3),(3,-3),(3,2),(0,2)$ onto the set R.



Its Jacobian is

$$\frac{\partial(g_1,g_2)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} \end{vmatrix} = -\frac{1}{5}$$

Therefore

$$\begin{aligned}
\iint\limits_R f(x,y)dA &= \iint\limits_W f\circ g(u,v)dA \\
&= \frac{1}{5}\int_0^3\int_{-3}^2 \left(\tfrac{1}{25}(v-u)(4v+u)+\tfrac{1}{5}(v-u)+\tfrac{1}{5}(4v+u)\right)dvdu \\
&= \frac{1}{125}\int_0^3\int_{-3}^2 \left(4v^2-3uv-u^2+5v\right)dvdu \\
&= \frac{1}{125}\int_0^3 \left[4v^2-3uv-u^2+5v\right]_{-3}^2 du = \frac{1}{25}\int_0^3\left[\frac{41}{6}+\frac{3u}{2}-u^2\right]_{-3}^2 du \\
&= \frac{1}{25}\left[\frac{41u}{6}+\frac{3u^2}{4}-\frac{u^3}{3}\right]_0^3 = \frac{1}{25}\left(\frac{41}{2}+\frac{27}{4}-9\right) = \frac{11}{60}
\end{aligned}$$

Exercise 5

1. *In each of the following problems, you are given a parallelogram R enclosed by four lines and a function f defined on R. You are required to (i) draw the parallelogram, (ii) determine a function g that maps a rectangle W, with sides parallel to the coordinate axes, onto R, and (iii) evaluate $\iint\limits_W f\circ g(u,v)\left|\frac{\partial(g_1,g_2)}{\partial(u,v)}\right|dA$.*

   (a) *R enclosed by the lines $y=2x$, $y=2x+7$, $y=-x-1$ and $y=-x+4$, and $f(x,y)=x+y^2$.*

   (b) *R enclosed by the lines $y=4x+2$, $y=4x-2$, $y=-x-2$ and $y=-x+1$, and $f(x,y)=x+xy+3y$.*

   (c) *R enclosed by the lines $y=3x+1$, $y=3x-5$, $y=-2x+2$ and $y=-2x-6$, and $f(x,y)=x^2+2y$*

   (d) *R enclosed by the lines $y=2x+5$, $y=2x-1$, $y=-2x-3$ and $y=-2x+5$, and $f(x,y)=x^2-3y^2$*

2. *Let R be the region inside the circle $x^2+y^2=9$ and outside the circle $x^2+y^2=1$. Determine a mapping that maps a rectangle W onto R then evaluate $\iint\limits_R xydA$.*

Another Look At Integrals In Polar Coordinates

We change variables from Cartesian to polar coordinates by composing a function $f(x, y)$, where x and y are Cartesian coordinates, with the function

$$g(r, \theta) = (g_1(r, \theta), g_2(r, \theta)) = (r \cos\theta, r \sin\theta)$$

By Theorem 2, to determine the integral of such a function $f(x, y)$ over a set R in the Cartesian plane, we must look for a the set W that g maps onto R then integrate $f \circ g(r, \theta) \left| \frac{\partial(g_1, g_2)}{\partial(r, \theta)} \right|$ over W. The Jacobian of the transformation is

$$\begin{vmatrix} \dfrac{\partial(r\cos\theta)}{\partial r} & \dfrac{\partial(r\cos\theta)}{\partial\theta} \\ \\ \dfrac{\partial(r\sin\theta)}{\partial r} & \dfrac{\partial(r\sin\theta)}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r$$

Therefore

$$\iint\limits_R f(x, y)\,dA = \iint\limits_W f(r\cos\theta, r\sin\theta) \left| \frac{\partial(g_1, g_2)}{\partial(r, \theta)} \right| dA = \iint\limits_W f(r\cos\theta, r\sin\theta) r\,dA. \qquad (1)$$