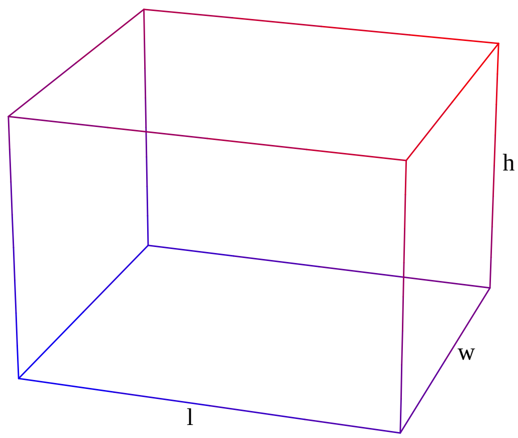
# Riemann Integral of a Function of Three Variables

A storage bin for ground flour has the shape of a rectangular box with dimensions $l$, $w$ and $h$ as shown below. Assume that the units are in meters. Say the bin is full of flour. If the density of the flour is constant and equal to $\rho$ kilogram per cubic meter then the weight of the flour in the bin would simply be the product of its volume and its density, i.e. it would be $lwh\rho$. Suppose the density is not constant, and it is $f(x, y, z)$ kilogram per cubic meter, at a point $(x, y, z)$. Then we have to resort to integration to calculate the exact weight of the flour. Assume that the storage bin is the set

$$B = \{(x, y, z) : 0 \leq x \leq l,\ 0 \leq y \leq w \text{ and } 0 \leq z \leq h\}$$




As you would expect, we have to partition $B$ into smaller elements, (boxes), and estimate the contribution of each element to the weight of the flour. Mimicking what we did in to calculate the volume of the solid enclosed by the graph of $f(x, y) = xy + 25$ and the rectangle with vertices at $(0, -2, 0)$, $(5, -2, 0)$, $(5, 4, 0)$, $(0, 4, 0)$ we do the following:

(a) Divide the interval $[0, l]$ into smaller subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$ where

$$0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = l.$$

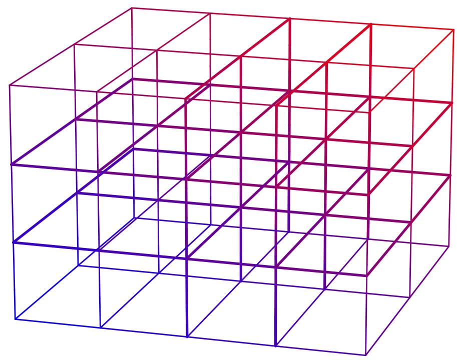
(b) Divide the interval $[0, w]$ into smaller subintervals $[y_0, y_1]$, $[y_1, y_2]$, $\ldots$, $[y_{m-1}, y_m]$ where

$$0 = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = w.$$

(c) Divide the interval $[0, h]$ into $t$ smaller subintervals $[z_0, z_1]$ , $[z_1, z_2]$, $\ldots$, $[z_{t-1}, z_t]$ where

$$0 = z_0 < z_1 < z_2 < \cdots < z_{\mathbf{t}-1} < z_{\mathbf{t}} = h.$$

Let $V_{ijk} = \{(x, y, z) : x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j \text{ and } z_{k-1} \leq z \leq z_k\}$. The $nmt$ boxes $V_{ijt}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $k = 1, \ldots, t$ partition $B$ into smaller rectangular boxes. (In the figure below, the given box is divided into $4 \times 2 \times 3 = 24$ smaller boxes.)



Let $\triangle x_i = (x_{i+1} - x_i)$, (the length of the interval $[x_i, x_{i+1}]$), $\triangle y_j = (y_{j+1} - y_j)$, (the length of the interval $[y_j, y_{j+1}]$), and $\triangle z_k = (z_{k+1} - z_k)$, (the length of the interval $[z_k, z_{k+1}]$). Then $V_{ijk}$ has volume

$$\triangle V_{ijk} = \triangle x_i \triangle y_j \triangle z_k$$

Let $(\theta_i, \alpha_j, \beta_k)$ be a point in $V_{ijk}$. Then $f(\theta_i, \alpha_j, \beta_k) \triangle V_{ijk}$ is an approximate value of the weight of the flour in the small box $V_{ijk}$ and the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{t} f(\theta_i, \alpha_j, \beta_k) \triangle V_{ijk} \tag{1}$$

is an approximate value of the weight of the flour in the storage bin. As you would expect, (1) is called a Riemann sum of $f$ determined by the boxes $V_{ijk}$. The limit of such sums as all the $\triangle x_i$'s, $\triangle y_j$'s and $\triangle z_k$'s tend to 0, (assuming the limit exists), should be the exact weight of the flour and it is called the Riemann integral of $f$ over $V$. It is denoted by

$$\iiint_B f(x, y, z) dV. \tag{2}$$

There are three integral signs because we took a limit of a triple sum, and the symbol $dV$ conveys the message that $B$ was partitioned into small elements with measure, (i.e. with volume),

$$\triangle V_{ijk} = \triangle x_i \triangle y_j \triangle z_k$$

.

If $f$ is continuous then we may, (like we have done with functions of two variables), calculate (2) using "partial integration". This time we start by keeping two of the variables constant and integrate the resulting function of one variable. We end up with a function of two variables which we handle as before. For example, suppose

$$f(x, y, z) = \frac{1 + x + y}{1 + z^2}.$$

If we keep $x$ and $y$ constant, (the constant value of $x$ should be between 0 and $l$ while the constant value of $y$ should be between 0 and $w$), we get a function

$$z \longrightarrow \frac{1 + x + y}{1 + z^2}$$

of one variable $z$ with with domain $[0, h]$. Its integral over the interval is a number that depends on the fixed values of $x$ and $y$,hence we may denote it by $u(x, y)$ and it is given by

$$u(x, y) = \int_0^h \left( \frac{1 + x + y}{1 + z^2} \right) dz = \left[ (\arctan z \, (1 + x + y)) \right]_0^h = (\arctan h) \, (1 + x + y)$$

Now we have a function $u(x,y) = (\arctan h)(1+x+y)$ of two variables $x$ and $y$ whose domain is the rectangle with corners at $(0,0,0)$, $(l,0,0)$, $(l,w,0)$ and $(0,w,0)$. If we keep $x$ constant between 0 and $l$ we get a function of one variable $y$ with domain $[0,w]$ and formula $y \to (\arctan h)(1+x+y)$. Its integral over the interval is a number that depends on the fixed values of $x$ hence we may denote it by $v(x)$ and it is given by

$$v(x) = \int_0^w (\arctan h)(1+x+y)\,dy = \left[(\arctan h)\left(y + yx + \tfrac{y^2}{2}\right)\right]_0^w = (\arctan h)\left(w + wx + \frac{w^2}{2}\right)$$

The function $v(x) = (\arctan h)\left(w + wx + \dfrac{w^2}{2}\right)$ of one variable $x$ has domain $[0,l]$. Its integral over the interval is

$$\int_0^l (\arctan h)\left(w + wx + \frac{w^2}{2}\right) dx = \left[(\arctan h)\left(wx + \frac{wx^2}{2} + \frac{w^2x}{2}\right)\right]_0^l = (\arctan h)\left(\frac{2wl + wl^2 + w^2l}{2}\right)$$

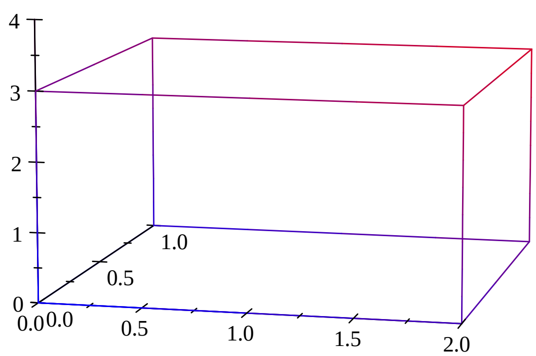
Fubini's theorem asserts that

$$\iiint_B \left(\frac{1+x+y}{1+z^2}\right) dV = (\arctan h)\left(\frac{2wl + wl^2 + w^2l}{2}\right)$$

The iterated integral may be written as

$$\int_0^l \left(\int_0^w \left(\int_0^h \left(\frac{1+x+y}{1+z^2}\right) dz\right) dy\right) dx \quad \text{or simply} \quad \int_0^l \int_0^w \int_0^h \left(\frac{1+x+y}{1+z^2}\right) dz dy dx$$

**Example 1** *Consider the set $B = \{(x,y,z) : 0 \le x \le 2,\ 0 \le y \le 1$ and $0 \le z \le 3\}$, (a box), and the function $f(x,y,z) = xyz^2 + 3y^2z$.*




*When we fix $z$ to a constant value between 0 and 3, and fix $y$ to a constant value between 0 and 1 we get a function of one variable $x$ with domain $[0,2]$. Its integral over the interval is a number which depends on the values of $z$ and $y$,hence we may denote it by $w(y,z)$ and it is given by*

$$w(y,z) = \int_0^2 \left(xyz^2 + 3y^2z\right) dx = \left[\frac{x^2yz^2}{2} + 3xy^2z\right]_0^2 = 2yz^2 + 6y^2z$$

*If we fix $z$ to a constant value between 0 and 3, then $w(y,z)$ gives us a function of one variable $y$ with domain $[0,1]$. Its integral over the interval is*

$$v(z) = \int_0^1 \left(2yz^2 + 6y^2z\right) dy = \left[y^2z^2 + 2y^3z\right]_0^1 = z^2 + 2z$$

*The domain of $v(z)$ is $[0,3]$ and its integral over the interval is*

$$\int_0^3 \left(z^2+2z\right)dz = \left[\frac{z^3}{3}+z^2\right]_0^3 = 18.$$

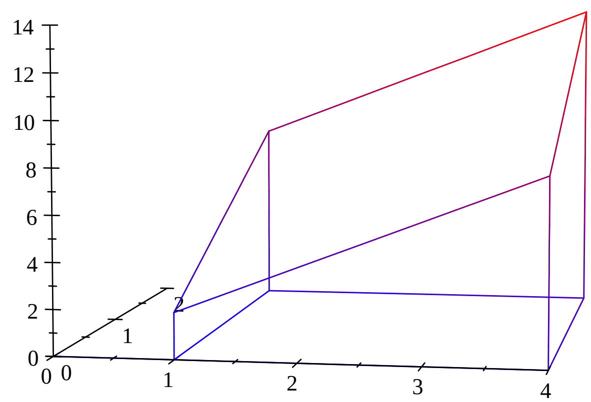
*Since we integrated with respect to $x$ then $y$ and finally $z$, we write this as*

$$\int_0^3\left(\int_0^1\left(\int_0^2 \left(xyz^2+3y^2z\right)dx\right)dy\right)dz = 18 \qquad \text{or simply} \qquad \int_0^3\int_0^1\int_0^2 \left(xyz^2+3y^2z\right)dxdydz = 18$$

*By Fubini's*

$$\iiint_B \left(xyz^2+3y^2z\right)dV = \int_0^3\int_0^1\int_0^2 \left(xyz^2+3y^2z\right)dxdydz = 18.$$

**Example 2** *Let $B$ be the set of points in space enclosed by the rectangle $\{(x,y,0): 1\le x\le 4 \text{ and } 0\le y\le 2\}$ and the plane $2x+3y-z=0$.*




*Let $f(x,y) = 4z+12xy$. To calculate its integral over $B$ using iterated integration, we keep $x$ fixed to a constant value between 1 and 4, and keep $y$ fixed to a constant value between 0 and 2 to get a function of one variable $z$ with domain $[0, 2x+3y]$. Its integral over the interval is a number that depends on $x$ and $y$ and is given by*

$$w(x,y) = \int_0^{2x+3y} (4z+12xy)\,dz = \left[2z^2+12xyz\right]_0^{2x+3y} = 8x^2+12xy+18y^2+24x^2y+36xy^2$$

*If we fix $x$ to a constant value between 1 and 4, the function $w(x,y)$ gives a function of one variable $y$ with domain $[0,2]$. Its integral over the interval is*

$$v(x) = \int_0^2 \left(8x^2+12xy+18y^2+24x^2y+36xy^2\right)dy = \tfrac{8}{3}x^3+60x^2+96x+48$$

*The domain of $v(x)$ is $[1,4]$ and its integral over the interval is*

$$\int_1^4 \left(\tfrac{8}{3}x^3+60x^2+96x+48\right)dx = \left[\tfrac{2}{3}x^4+20x^3+48x^2+48x\right]_1^4 = 1194$$

*Therefore* $\displaystyle\iiint_B (4z+12xy)\,dV = 2294$

**Exercise 3** *Evaluate each given iterated integral*

1. $\int_{-1}^{2}\int_{0}^{4}\int_{1}^{3} xyz^2 dzdydx$

2. $\int_{0}^{2}\int_{-3}^{4}\int_{1}^{5} (x+2y-3z)dxdydz$

3. $\int_{0}^{2}\int_{0}^{x}\int_{1}^{x+y} xyzdzdydx$

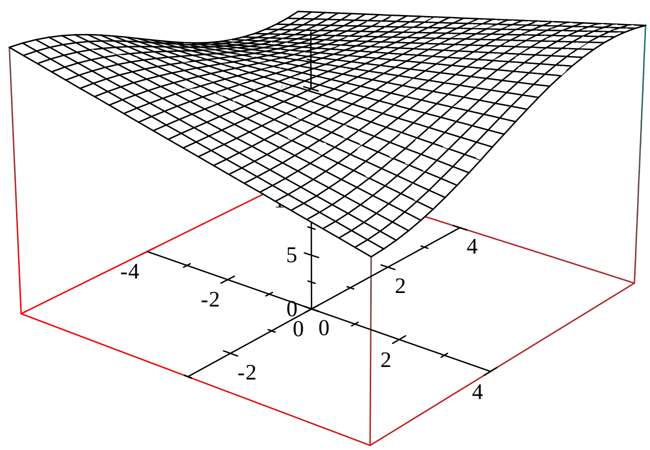
4. $\int_{0}^{4}\int_{-1}^{2}\int_{0}^{2} \left(x^2y+y^2-z^2\right) dxdydz$

## An Intuitive Proof of Fubini's Theorem for a Function of Two Variables

Instead of giving a general proof, we outline it through an example. To this end, consider the function $f(x,y) = 20 + x\sin\frac{1}{2}y$ and the rectangle

$$R = \{(x,y) : -4 \leq x \leq 4 \text{ and } -3 \leq y \leq 4\}$$

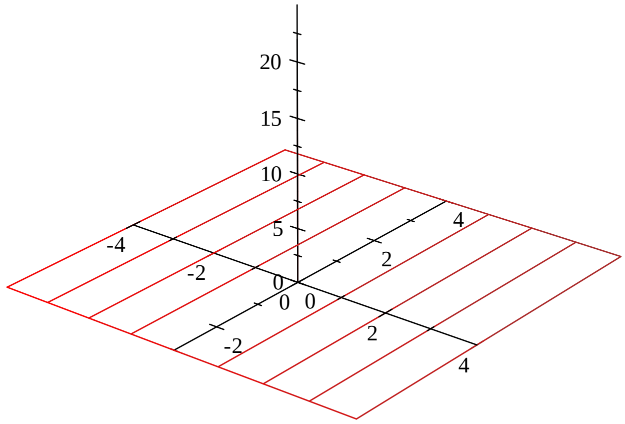
As we have already pointed out, $\iint\limits_R f(x,y)dA$ may be viewed as the volume of the solid, shown below, enclosed by the graph of $f$ and the rectangle $R$



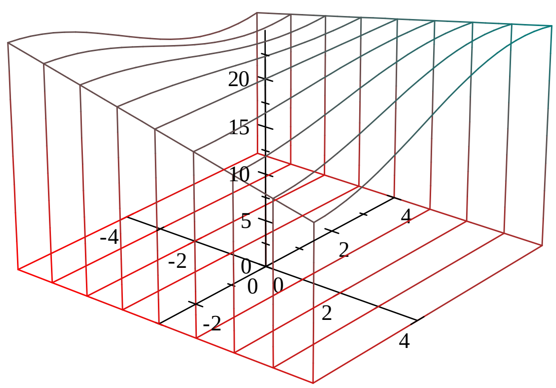

To estimate this volume, we partition the solid into thin strips as follows: Divide the interval $[-4,4]$ into smaller subintervals $[x_0,x_1]$, $[x_1,x_2]$, ..., $[x_{n-1},x_n]$ where

$$-4 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 4$$

For simplicity, we may assume that they all have the same length $\triangle x = 8 \div n$. They partition the rectangle $R$ into $n$ thin rectangles which in turn partition the solid into $n$ thin strips. In the figures below, the rectangle was partitioned into 8 smaller rectangles which, in turn, partitioned the solid into 8 strips.

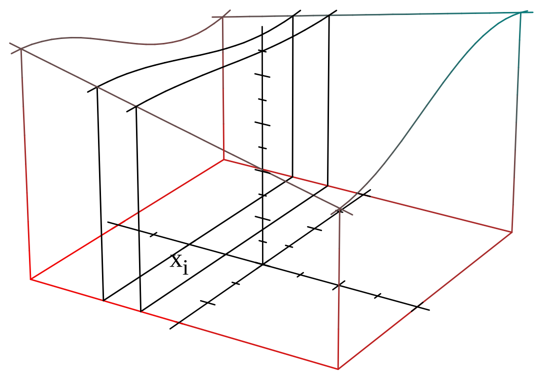



Rectangle $R$ partitioned into 8 thinner rectangles



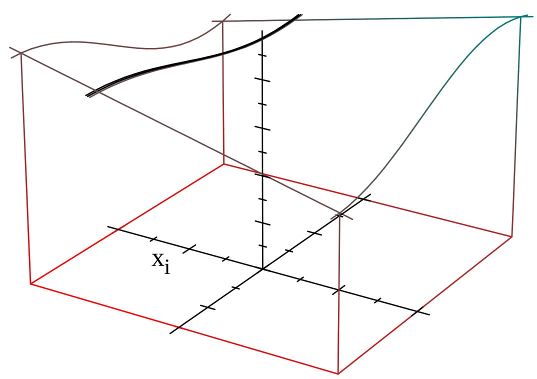

Solid partitioned into 8 thinner strips

Take a typical strip, shown below, enclosed by the graph of $f$ and the rectangle with vertices at $(x_i, -3, 0)$, $(x_{i+1}, -3, 0)$, $(x_{i+1}, 4, 0)$ and $(x_i, 4, 0)$.



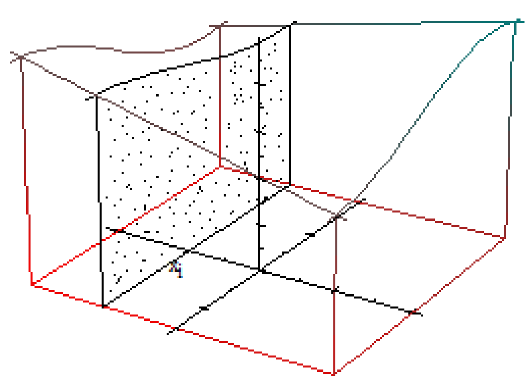

Typical strip

The graph of $f(x_i, y)$, $-3 \le y \le 4$ is shown below.




Graph of $f(x_i, y)$, $-3 \le y \le 4$, highlighted in the graph of $f$

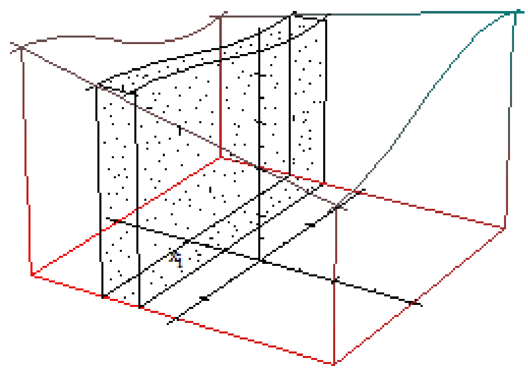
In the next figure, the region enclosed by the graph of $f(x_i, y)$ and the line segment joining $(x_i, -3, 0)$ and $(x_i, 4, 0)$ is shaded.



Its area is

$$\int_{-3}^{4} f(x_i, y)dy$$

It shouldn't be hard to convince you that the number $\left(\int_{-3}^{4} f(x_i, y)dy\right) \triangle x$, which is the volume of the shaded strip below, is a good approximation to the volume of the typical strip.



Therefore

$$\sum_{i=0}^{n-1} \left(\int_{-3}^{4} f(x_i, y)dy\right) \triangle x \simeq \text{ Volume of the solid.}$$

The approximations should improve as $\triangle x \to 0$. Therefore

$$\lim_{\triangle x \to 0} \sum_{i=0}^{n-1} \left(\int_{-3}^{4} f(x_i, y)dy\right) \triangle x = \text{ Volume of the solid.}$$

The limit of the sums is $\int_{-4}^{4} \left(\int_{-3}^{4} f(x, y)dy\right) dx$. It follows that

$$\int_{-4}^{4} \left(\int_{-3}^{4} f(x, y)dy\right) dx = \iint_{R} f(x, y)dA$$

**Exercise 4** *Show, in a similar way, that*

$$\int_{-3}^{4} \left(\int_{-4}^{4} f(x, y)dx\right) dy = \iint_{R} f(x, y)dA$$

*Give all the essential details.*