Evaluating a Double Integral by Iterations

For a continuous function f and regions R enclosed by continuous curves, we may calculate the double integral $\iint_R f(x,y)dA$ using what one may call "partial integration" and thus avoid the process of forming Riemann sums. The process involves keeping one of the two variables constant, (we did this when computing partial derivatives), and integrate the resulting function of one variable. We illustrate the process using the integral of $f(x,y) = 25+xy$ over the rectangle R with vertices at $(0,-2,0)$, $(5,-2,0)$, $(5,4,0)$, $(0,4,0)$ which we evaluated using the limits of sums. If we keep x constant, (the constant value must be between 0 and 5, else there would be no numbers y such that $(x,y) \in R$), we get a function $y \to xy+25$ of one variable y with domain $[-2,4]$. Its integral over $[-2,4]$ is

$$\int_{-2}^{4} (25+xy)\,dy = \left[10y + \tfrac{1}{2}xy^2\right]_{-2}^{4} = (100+8x) - (-50+2x) = 150+6x$$

We now have a new a function $u(x) = 150+6x$ of one variable x with domain $[0,5]$. Its integral over its domain $[0,5]$ is

$$\int_{0}^{5} (150+6x)\,dx = \left[150x + 3x^2\right]_{0}^{5} = 825$$

which is the value of the double integral $\iint_R f(x,y)dA$.

We could have kept y constant, (the constant value must be between -2 and 4), to get a function $x \to xy+25$ of one variable x with domain $[0,5]$. Its integral over this set is

$$\int_{0}^{5} (xy+25)\,dx = \left[25x + \tfrac{1}{2}yx^2\right]_{0}^{5} = 125 + \tfrac{25}{2}y$$

which is a function $v(y) = 125 + \frac{25}{2}y$ of one variable y with domain $[-2,4]$. When we integrate it over this set and the result is

$$\int_{-2}^{4} \left(125 + \tfrac{25}{2}y\right) dy = \left[125y + \tfrac{25}{4}y^2\right]_{-2}^{4} = (500+100) - (-250+25) = 825.$$

Since $u(x) = \int_{-2}^{4} (25+xy)\,dy$, the integral of u over $[0,5]$ is denoted by

$$\int_0^5 \left(\int_{-2}^{4} (25+xy)\,dy\right) dx \qquad \text{or simply} \qquad \int_0^5 \int_{-2}^{4} (25+xy)\,dydx$$

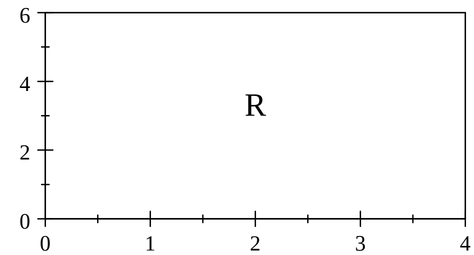
and it is called an iterated integral of f over R. Likewise, $v(y) = \int_0^5 (xy+25)\,dx$, therefore its integral over $[-2,4]$ is denoted by

$$\int_{-2}^{4} \left(\int_{0}^{5} (xy+25)\,dx\right) dy \qquad \text{or simply} \qquad \int_{-2}^{4} \int_{0}^{5} (xy+25)\,dxdy$$

and is also called an iterated integral of f over R. Note that in an iterated integral, the order of the symbols dx and dy matters. When they appear as $dydx$ the instruction is to keep x fixed and integrate a function of one variable y with respect to y. The result will be a function of one variable x which you would then proceed to integrate with respect to x. When they appear as $dxdy$, we first fix y.

We verified directly that the two iterated integrals $\int_0^5 \int_{-2}^{4} (25+xy)\,dydx$ and $\int_{-2}^{4} \int_{0}^{5} (xy+25)\,dxdy$ are equal to the Riemann integral of f over R. In general, if f is a continuous function and R is a set enclosed by continuous curves then the Riemann integral of f over R is equal to any one of its iterated integral over R. This statement is called **Fubini's theorem**.

Example 1 *Let $f(x,y) = x^2y + y^3 - 3x$ and R be the rectangle with vertices at $(0,0)$, $(4,0)$, $(4,6)$ and $(0,6)$.*



If we keep the first variable x constant, we get a function of one variable y with domain $[0,6]$. Its integral is

$$u(x) = \int_0^6 \left(x^2y + y^3 - 3x\right) dy = \left[\tfrac{1}{2}x^2y^2 + \tfrac{1}{4}y^4 - 3xy\right]_0^6 = 18x^2 - 18x + 324$$

The domain of u is $[0,4]$, because x can be assigned any constant value in the interval $[0,4]$, and its integral on this interval is

$$\int_0^4 \left(18x^2 - 18x + 324\right) dx = \left[6x^3 - 9x^2 + 324x\right]_0^4 = 384 - 144 + 1296 = 1536$$

By Fubini's theorem,

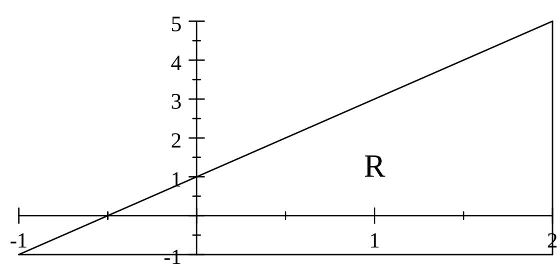
$$\iint_R \left(x^2y + y^3 - 3x\right) dA = \int_0^4 \int_0^6 \left(x^2y + y^3 - 3x\right) dydx = 1536$$

The other iterated integral is

$$\begin{aligned}\int_0^6 \int_0^4 \left(x^2y + y^3 - 3x\right) dxdy &= \int_0^6 \left(\left[\tfrac{1}{3}x^3y + xy^3 - \tfrac{3}{2}x^2\right]_0^4\right) dy = \int_0^6 \left(\tfrac{64}{3}y + 4y^3 - 24\right) dy \\ &= \left[\tfrac{32}{3}y^2 + y^4 - 24y\right]_0^6 = 384 + 1296 - 144 = 1536\end{aligned}$$

The region R does not have to be a rectangle:

Example 2 *Let $f(x,y) = 12 + x - y$ and R be the triangle enclosed by lines $y = 2x + 1$, $y = -1$ and $x = 2$.*



Its vertices are at $(-1,-1)$, $(2,-1)$ and $(2,5)$. Let $f(x,y) = 12 + x - y$. If we fix x to a constant value between -1 and 2, we get a function of one variable y with domain $[-1, 2x+1]$. Its integral over the interval is

$$\begin{aligned}u(x) &= \int_{-2}^{2x+1} (12 + x - y)\, dy = \left[12y + xy - \tfrac{1}{2}y^2\right]_{-1}^{2x+1} \\ &= 12\,(2x+1) + x\,(2x+1) - \tfrac{1}{4}\left(4x^2 + 4x + 1\right) - \left(-12 - x - \tfrac{1}{2}\right) = 24x + 24\end{aligned}$$

The integral of $u(x) = 24x + 24$ *over* $[-1, 2]$ *is*

$$\int_{-1}^{2} (24x + 24)\, dx = \left[12x^2 + 24x\right]_{-1}^{2} = 48 + 48 - 12 + 24 = 108$$

Thus

$$\iint\limits_R (12 + x - y)\, dA = \int_{-1}^{2} \int_{-2}^{2x+1} (12 + x - y)\, dydx = 108$$

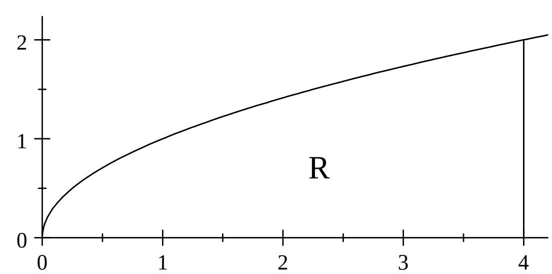
Say you choose to keep y*, (instead of* x*), constant. The constant value of* y *must be between* -1 *and* 5*. You get a function of one variable with domain* $\left[\frac{1}{2}(y-1), 2\right]$*. Its integral over this set is*

$$\begin{aligned}
\int_{\frac{1}{2}(y-1)}^{2} (12 + x - y)\, dx &= \left[12x + \tfrac{1}{2}x^2 - xy\right]_{\frac{1}{2}(y-1)}^{2} \\
&= 2x + 2 - 2y - \left[12 \cdot \tfrac{1}{2}(y-1) + \tfrac{1}{8}(y-1)^2 - \tfrac{1}{2}(y-1)y\right] \\
&= \frac{255}{8} - \frac{33y}{4} + \frac{3y^2}{8}
\end{aligned}$$

The integral of $v(y) = \dfrac{255}{8} - \dfrac{33y}{4} + \dfrac{3y^2}{8}$ *over* $[-1, 5]$ *is*

$$\begin{aligned}
\int_{-1}^{5} \left(\frac{255}{8} - \frac{33y}{4} + \frac{3y^2}{8}\right) dy &= \left[\frac{255y}{8} - \frac{33y^2}{8} + \frac{y^3}{8}\right]_{-1}^{5} \\
&= \frac{1275 - 825 + 125}{8} - \frac{-255 - 33 - 1}{8} = \frac{864}{8} = 108
\end{aligned}$$

Example 3 *Let* R *be the region in the plane enclosed by the curve* $y = \sqrt{x}$*, the* x*-axis and the line* $x = 4$*.*



Let $f(x, y) = 2x + y + 3xy$*. To evaluate its iterated integrals over* R*, we observe first, that if we choose a fixed value of* x *between* 0 *and* 4*, (we cannot choose values of* x *outside this interval because there would be no numbers* y *such that* $(x, y) \in R$*), we get a function of one variable* y *with domain* $[0, \sqrt{x}]$*. Its integral over this set is*

$$u(x) = \int_{0}^{\sqrt{x}} (2x + y + 3xy)\, dy = \left[2xy + \tfrac{1}{2}y^2 + \tfrac{3}{2}xy^2\right]_0^{\sqrt{x}} = 2x^{3/2} + \tfrac{1}{2}x + \tfrac{3}{2}x^2$$

The domain of u *is the interval* $[0, 4]$ *and*

$$\int_0^4 u(x)dx = \int_0^4 \left(2x^{3/2} + \tfrac{1}{2}x + \tfrac{3}{2}x^2\right) dx = \left[\tfrac{4}{5}x^{5/2} + \tfrac{1}{4}x^2 + \tfrac{1}{2}x^3\right]_0^4 = \frac{308}{5}.$$

By Fubini's theorem,

$$\iint\limits_R (2x + y + 3xy)\, dA = \int_0^4 \int_0^{\sqrt{x}} (2x + y + 3xy)\, dydx = \frac{308}{5}$$

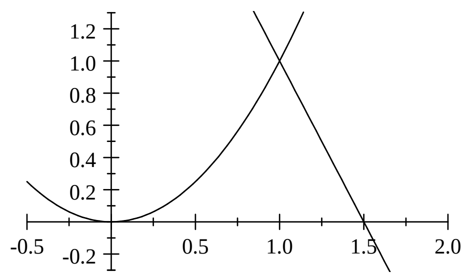
To evaluate the other iterated integral, note that we have to choose a fixed value of y *between* 0 *and* 2*. Then we get a function of one variable* x *with domain* $\left[y^2, 4\right]$ *and its integral over the interval is*

$$v(y) = \int_{y^2}^{4} (2x + y + 3xy)\, dx = \left[x^2 + xy + \tfrac{3}{2}x^2 y\right]_{y^2}^{4} = 16 + 28y - y^4 - y^3 - \tfrac{3}{2}y^5$$

The domain of v *is the interval* $[0, 2]$ *and its integral over this interval is*

$$\int_0^2 \left(16 + 28y - y^4 - y^3 - \tfrac{3}{2}y^5\right) dy = \left[16y + 14y^2 - \tfrac{1}{5}y^5 - \tfrac{1}{4}y^4 - \tfrac{1}{4}y^6\right]_0^2 = \frac{308}{5}.$$

Example 4 *Let* R *be the set in the first quadrant enclosed by the parabola* $y = x^2$*, the line* $y = -2x + 3$ *and the* x*-axis. The parabola intersects the line, (in the first quadrant), at* $(1, 1)$*.*



Say we have to evaluate the integral of $f(x, y) = 4x + y$ *over* R*. If we choose to fix* x *and vary* y*, we must evaluate two different integrals because: (i) when* $0 \leq x \leq 1$*,* y *varies from 0 to* x^2*, and (ii) when* $1 \leq x \leq \frac{3}{2}$*,* y *varies from 0 to* $3 - 2x$*. Thus*

$$\begin{aligned}
\iint\limits_R (4x + y)\, dA &= \int_0^1 \int_0^{x^2} (4x + y)\, dy dx + \int_1^{1.5} \int_0^{3-2x} (4x + y)\, dy dx \\
&= \int_0^1 \left(\left[4xy + \tfrac{1}{2}y^2\right]_0^{x^2}\right) dx + \int_1^{1.5} \left(\left[4xy + \tfrac{1}{2}y^2\right]_0^{3-2x}\right) dx \\
&= \int_0^1 \left(4x^3 + \tfrac{1}{2}x^4\right) dx + \int_1^{1.5} \left(\tfrac{9}{2} + 6x - 6x^2\right) dx \\
&= \left[x^4 + \frac{x^5}{10}\right]_0^1 + \left[\frac{9}{2}x + 3x^2 - 2x^3\right]_1^{1.5} = \frac{47}{20}
\end{aligned}$$

However, if we choose to keep y *fixed and vary* x*, we end up with one integral, since* x *varies from* $\sqrt{y}$ *to* $\frac{1}{2}(3 - y)$*. Indeed*

$$\begin{aligned}
\iint\limits_R (4x + y)\, dA &= \int_0^1 \int_{\sqrt{y}}^{\frac{1}{2}(3-y)} (4x + y)\, dx dy = \int_0^1 \left(\left[2x^2 + xy\right]_{\sqrt{y}}^{\frac{1}{2}(3-y)}\right) dy \\
&= \int_0^1 \left(\frac{9}{2} - \frac{7y}{2} - y^{3/2}\right) dy = \left[\frac{9y}{2} - \frac{7y^2}{4} - \frac{2}{5}y^{5/2}\right]_0^1 = \frac{47}{20}
\end{aligned}$$

Exercise 5

1. *Evaluate each iterated integral*

(a) $\int_0^1 \int_{-1}^2 \left(x^2 + 3xy - y^3\right) dxdy$

(b) $\int_1^2 \int_3^5 \left(\frac{3}{x} + 6y - \frac{4x}{y}\right) dydx$

(c) $\int_1^2 \int_0^{1/2} (\sin \pi y + 6y - \cos 2x)\, dydx$

(d) $\int_1^2 \int_0^1 \left(e^{2x} + 2xy\right) dxdy$

(e) $\int_1^2 \int_0^x (e^y + x + y)\, dydx$

(f) $\int_0^3 \int_0^x \left(8x - 4y + x^2y^3\right) dydx$

(g) $\int_0^4 \int_0^{\frac{1}{2}y} (xe^y - 4xy + 1)\, dxdy$

(h) $\int_0^4 \int_{-y}^{2y} (x + y - 6xy)\, dxdy$

2. *Consider the iterated integral* $\int_0^1 \int_{-1}^2 \left(x^2 + 3xy - y^3\right) dxdy$ *in question 1 (a) above. Note that we keep y constant between 0 and 1 then integrate a function of one variable x on the interval* $[-1, 2]$*. Therefore we integrate* $\left(x^2 + 3xy - y^3\right)$ *over a region consisting of all values of x between* -1 *and* 2 *and all values of y between* 0 *and* 1*. The region must be the rectangle with vertices at* $(-1, 0)$*,* $(2, 0)$*,* $(2, 1)$ *and* $(-1, 1)$*. In the case of* $\int_0^3 \int_0^x \left(8x - 4y + x^2y^3\right) dydx$ *in part (f) of the same question, we keep x constant between 0 and 3 and integrate the resulting function of one variable y over the interval* $[0, x]$*. Therefore we integrate* $\left(x^2 + 3xy - y^3\right)$ *over a region consisting of all values of x between* 0 *and* 3 *and all values of y between the line* $y = 0$ *and the line* $y = x$*. The region must be the triangle with vertices at* $(0, 0)$*,* $(3, 0)$ *and* $(3, 3)$*. Determine the region of integration in parts (d), (e) and (h).*

3. *Let* $f(x, y) = 4x - 6xy$ *and R be the region in the first quadrant enclosed by the line* $y = x$ *and the curve* $y = \sqrt{x}$*. Show that* $\iint\limits_R f(x, y)dA = \frac{1}{60}$.

4. *Determine* $\iint\limits_R (x + 4xy)\, dA$ *where R is the region in the first quadrant enclosed by the parabola* $y = x^2$*, the line* $x = 2$*, and the x-axis.*

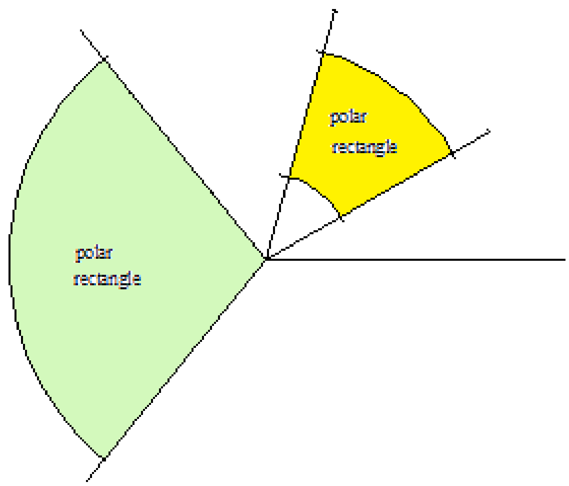
5. *Determine* $\iint\limits_R \left(x^2 + 4xy - y^2\right) dA$ *where R is the region enclosed by the curve* $y = x^{1/2}$*, the line* $y = 2$*, and the y-axis.*

6. *Evaluate the integral of* $f(x, y) = x + 2y$ *over the set R enclosed by the parabola* $y = 3x - x^2$ *and the line* $y = 2x$*.*

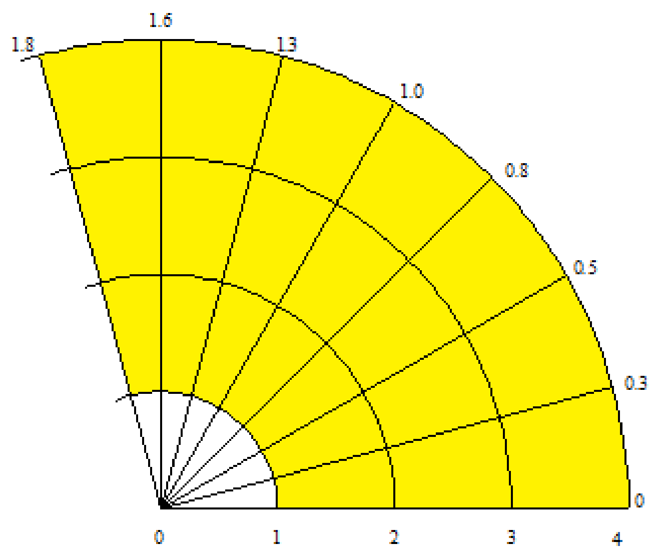
7. *Let R be a region in the x - y plane and f be the constant function* $f(x, y) = 1$*. Show that* $\iint\limits_R f(x, y)dA$ *is the area of the region R.*

Double Integral in Polar Coordinates

Let a, b be nonnegative real numbers with $a < b$, and α, β be real numbers such that $\alpha < \beta$ and $\beta - \alpha \leq 2\pi$. The set $\{(r, \theta) : a \leq r \leq b \text{ and } \alpha \leq \theta \leq \beta\}$ is called a polar rectangle. In the figure below, the polar rectangles $\{(r, \theta) : 1 \leq r \leq 2.5 \text{ and } 0.5 \leq \theta \leq 1.3\}$ and $\{(r, \theta) : 0 \leq r \leq 3 \text{ and } 2.2 \leq \theta \leq 4.0\}$ are shaded.



A polar rectangle $\{(r,\theta) : a \le r \le b \text{ and } \alpha \le \theta \le \beta\}$ may be partitioned into smaller polar rectangles by partitioning the intervals $[a, b]$ and $[\alpha, \beta]$ into smaller subintervals. In the figure below, the polar rectangle $\{(r,\theta) : 1 \le r \le 4 \text{ and } 0 \le \theta \le 1.8\}$ is partitioned into 21 smaller polar rectangles by partitioning $[1, 4]$ into subintervals $[1, 2]$, $[2, 3]$ and $[3, 4]$, and partitioning $[0, 1.8]$ into subintervals $[0, 0.3]$, $[0.3, 0.5]$, $[0.5, 0.8]$, $[0.8, 1]$, $[1, 1.3]$, $[1.3, 1.6]$ and $[1.6, 1.8]$.



A polar rectangle partitioned into smaller polar rectangles

Let $W = \{(r,\theta) : a \le r \le b \text{ and } \alpha \le \theta \le \beta\}$ be a polar rectangle and $f(x, y)$ a function defined on W whose formula is expressed in Cartesian coordinates x and y. The relation between (x, y) and (r, θ) is $x = r\cos\theta$ and $y = r\sin\theta$. Partition the interval $[a, b]$ into smaller subintervals $[r_0, r_1]$, $[r_1, r_2]$, $\ldots$, $[r_{n-1}, r_n]$ where $a = r_0 < r_1 < \cdots < r_{n-1} < r_n = b$, and partition $[\alpha, \beta]$ into smaller subintervals $[\theta_0, \theta_1]$, $[\theta_1, \theta_2]$, $\ldots$, $[\theta_{m-1}, \theta_m]$ with $\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{m-1} < \theta_m = \beta$. Let $\triangle r_i = (r_{i+1} - r_i)$, (the length of the ith interval $[r_i, r_{i+1}]$) and $\triangle\theta_j = (\theta_{j+1} - \theta_j)$, (the length of the jth interval $[\theta_j, \theta_{j+1}]$. The subintervals determine smaller polar rectangles $W_{ij} = \{(r, \theta) : < r_{i-1} \le r \le r_i \text{ and } \theta_{j-1} \le \theta \le \theta_j\}$, $i = 1, \ldots, n$; $j = 1, \ldots, m$. Denote the area of W_{ij} by $\triangle W_{ij}$. Recall that in a circle of radius r, a sector subtended by an angle of θ radians has area $\frac{1}{2}r^2\theta$. Therefore

$$\triangle W_{ij} = \frac{1}{2}\left[r_i^2\left(\theta_j - \theta_{j-1}\right) - r_{i-1}^2\left(\theta_j - \theta_{j-1}\right)\right] = \frac{1}{2}\left[r_i^2 - r_{i-1}^2\right]\left(\theta_j - \theta_{j-1}\right) = \frac{1}{2}\left(r_i + r_{i-1}\right)\left(r_i - r_{i-1}\right)\left(\theta_j - \theta_{j-1}\right)$$

When the subintervals $[r_i, r_{i+1}]$ are small, $\frac{1}{2}(r_i + r_{i-1})$ may be approximated by $\frac{1}{2}(r_i + r_i) = r_i$. Therefore

$$\triangle W_{ij} \simeq r_i (r_i - r_{i-1})(\theta_j - \theta_{j-1}) = r_i (\triangle r_i \triangle \theta_j)$$

In the Cartesian coordinate system, we denoted the products $\triangle x_i \triangle y_j$ by $\triangle A_{ij}$. We adopt the same notation here and denote $r_i (\triangle r_i \triangle \theta_j)$ by $r_i \triangle A_{ij}$. We must pick a point (r, θ) from each element W_{ij}. To simplify notation pick (r_i, θ_j). We then form the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i \cos\theta_j, r_i \sin\theta_j) \triangle W_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} f(r_i \cos\theta_j, r_i \sin\theta_j) r_i \triangle A_{ij}$$

The limit of such sums as all the $\triangle r_i$'s and $\triangle \theta_j$'s shrink to 0 is called the integral of f over W in polar coordinates. We will denote it by

$$\iint_W f(r\cos\theta, r\sin\theta) r dA \tag{1}$$

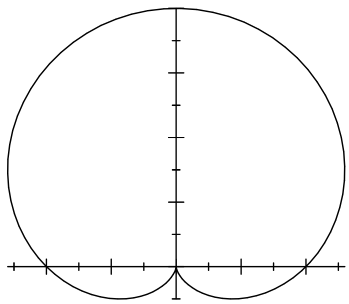
We use two integral signs because it is a limit of a double summation and the symbol rdA points out that the area of an element W_{ij} is $r_i \triangle A_{ij} = r_i \triangle r_i \triangle \theta_j$. The following examples show how we may evaluate $\iint_W f(r\cos\theta, r\sin\theta) r dA$ using iterated integrals.

Example 6 *Let $f(x, y) = x^2 + y + 4$ and W be the region enclosed by the circles centered at $(0, 0)$ with radius 2 and 4 respectively. Thus W is the polar rectangle $\{(r, \theta) : 2 \le r \le 4$ and $0 \le \theta \le 2\pi\}$. Clearly, $f(r\cos\theta, r\sin\theta) = r^2\cos^2\theta + r\sin\theta + 5$. To evaluate $\iint_W \left(r^2\cos^2\theta + r\sin\theta + 5\right) rdA$ by iteration, note that if we fix θ, then r varies from 2 to 4. Since the fixed values of θ must be chosen from $[0, 2\pi]$,*

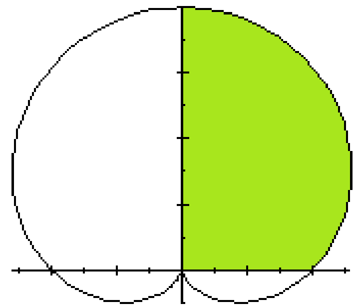
$$\begin{aligned}
\iint_W \left(r^2\cos^2\theta + r\sin\theta + 5\right) rdA &= \int_0^{2\pi} \int_2^4 \left(r^3\cos^2\theta + r^2\sin\theta + 5r\right) drd\theta \\
&= \int_0^{2\pi} \left[\frac{r^4}{4}\cos^2\theta + \frac{r^3}{3}\sin\theta + \frac{5r^2}{2}\right]_2^4 d\theta \\
&= \int_0^{2\pi} \left(60\cos^2\theta + \frac{56}{3}\sin\theta + 30\right) d\theta \\
&= \int_0^{2\pi} \left(60 + 30\cos 2\theta + \frac{56}{3}\sin\theta\right) d\theta = 120\pi
\end{aligned}$$

Example 7 *The curve $r = 1 + \sin\theta$, $0 \le \theta \le 2\pi$, (called a cardioid because it looks like a heart), is shown below. The shaded region is the set W of points (r, θ) such that $0 \le r \le 1 + \sin\theta$ and $0 \le \theta \le \frac{1}{2}\pi$. Let*

$f(x,y) = xy$. *The integral of* f *over* W, *using polar coordinates, is* $\iint\limits_W \left(r^2\cos\theta\sin\theta\right) r dr d\theta$



A cardioid



The set W

To evaluate the integral by iteration, note that if we fix θ *between 0 and* $\frac{1}{2}\pi$ *then* r *varies from* 0 *to* $1+\sin\theta$. *Therefore*

$$\begin{aligned}\iint\limits_W \left(r^2\cos\theta\sin\theta\right) r dA &= \int_0^{\frac{1}{2}\pi}\int_0^{1+\sin\theta} r^3\cos\theta\sin\theta dr d\theta = \int_0^{\frac{1}{2}\pi}\left(\left[\frac{r^4}{4}\right]_0^{1+\sin\theta}\right)\cos\theta\sin\theta d\theta \\ &= \tfrac{1}{4}\int_0^{\frac{1}{2}\pi}(1+\sin\theta)^4\sin\theta\cos\theta d\theta\end{aligned}$$

Make a substitution $u = 1+\sin\theta$ *and deduce that* $\frac{1}{4}\int_0^{\frac{1}{2}\pi}(1+\sin\theta)^4\sin\theta\cos\theta d\theta = \dfrac{43}{40}$

Remark 8 *Integrals in polar coordinates may be handled under the general topic "changing variables in an integral". For more details, go to page* **??**.

Exercise 9

1. *Let* R *be the region in the upper half of the plane enclosed by the* x*-axis and the curve* $x^2+y^2=4$. *Change to polar coordinates and evaluate* $\iint\limits_R \sin\sqrt{x^2+y^2} dA$.

2. *Let* R *be the set of points below the circle* $x^2+y^2=1$ *and above the line* $y=x$. *Let* $f(x,y) = x^2+y^2+6x$. *Convert to polar coordinates and evaluate* $\iint\limits_R \left(x^2+y^2+6x\right) dA$.

3. *Consider the curve with polar equation* $r = 4\sin\theta$. *If we change to Cartesian coordinates we obtain, (because* $r=\sqrt{x^2+y^2}$ *and* $\sin\theta = \dfrac{y}{r}$*)*

$$\sqrt{x^2+y^2} = \frac{4y}{\sqrt{x^2+y^2}}$$

which may be transformed into $x^2+y^2-4y=0$. *Completing squares, we obtain* $x^2+(y-2)^2 = 4$ *which reveals that the curve is a circle centred at* $(0,2)$ *with radius 2. Note that the entire circle is traced out when* θ *varies from* 0 *to* π. *Let* R *be the region enclosed by this circle and* $f(x,y) = \dfrac{1}{x^2+y^2+1}$.

Convert to polar coordinates and evaluate $\iint\limits_R \dfrac{dA}{x^2+y^2+1}$.

4. Let R be the region enclosed by the curve with polar equation $r = 6\cos\theta$ and $f(x,y) = \sqrt{x^2+y^2+5}$.

Evaluate $\displaystyle\iint_R f(x,y)\,dA$.