Integrals of Functions of Several Variables

We often resort to integrations in order to determine the exact value I of some quantity which we are unable to evaluate by performing a finite number of addition or multiplication operations. For example, we have to evaluate an integral to find the area of the region between the graph of $f(x) = x^2 + 3$ and the interval $[0, 2]$. The integral is

$$\int_0^2 \left(x^2 + 3\right) dx \tag{1}$$

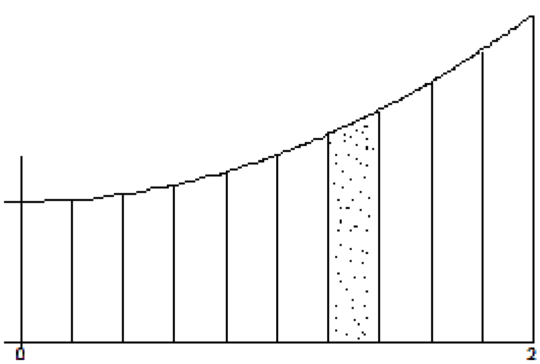
We cannot determine this area by calculating the areas of a finite number of rectangles or triangles because the region is not entirely enclosed by straight line segments.

In general, an integration problem involves the following basic items:

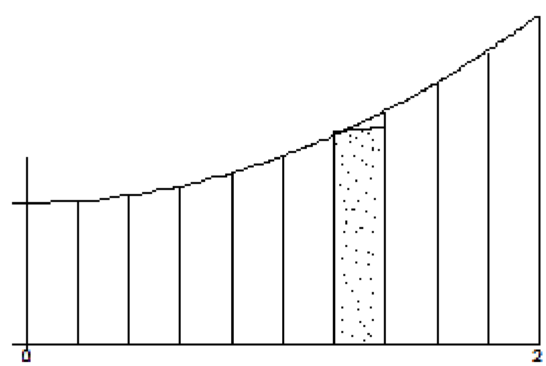
1. Some quantity I whose exact value we must determine, but we cannot calculate it by performing a finite number of additions/multiplications.

2. Some set S that has a length, if it is a subset of the real line $\mathbb{R}$, an area if it is a subset of the plane, or a volume if it is a subset of 3-dimencsional space. A general term for the length or the area or the volume of a set S is a *measure of* S. It should be possible to divide S into a finite number of "smaller segments" $S_1, S_2, \ldots, S_n$, to be called "elements of S", which also have measures. In the integral (1), the set S is the interval $[0, 2]$, with length 2. We can subdivide it into smaller subintervals, e.g. n equal subintervals of length $\triangle l = \frac{2}{n}$ each. The symbol $\triangle l$ is pronounced "delta l" which you should think of as a "small length".

3. Some function f defined on S such that the product of the measure of an element S_i and the value of f at some point $x_i \in S_i$ gives a reasonable estimate of the contribution of the element S_i to the exact value I we wish to determine. In the integral (1), the function is $f(x) = x^2 + 3$. If we divide $S = [0, 2]$ into n smaller subintervals of length $\triangle l = \frac{2}{n}$ each and pick a point x_i in an element $S_i = \left[\frac{2i}{n}, \frac{2(i+1)}{n}\right]$ then the product $f(x_i) \cdot \triangle l$ gives an approximate value of the area enclosed by the segment S_i and the graph of f.



Area between segment S_i and the graph of f.



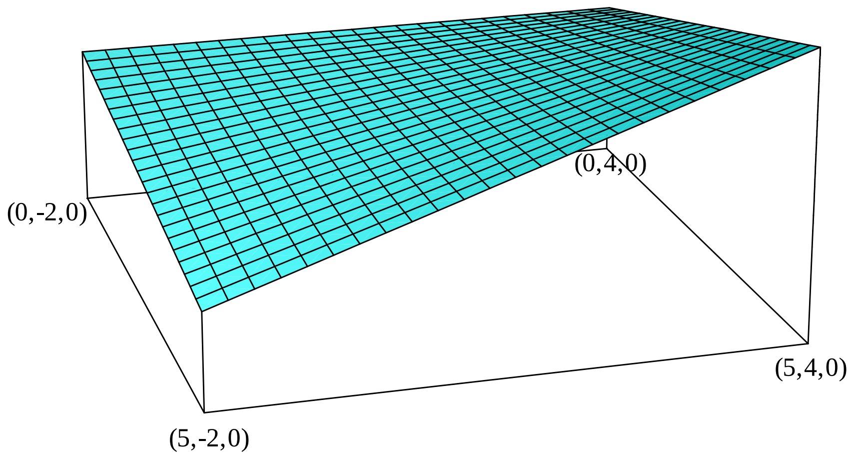
Approximating area

Evaluating the integral boils down to doing the following:

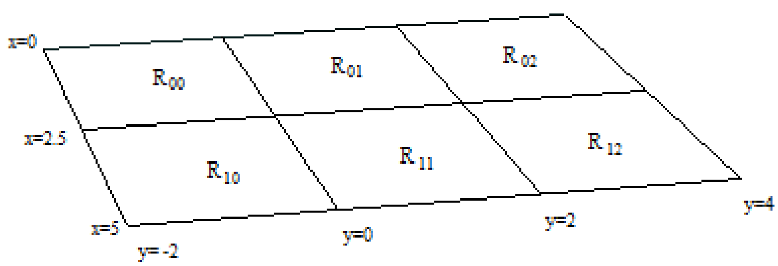
- Slice S into a finite number of smaller elements S_i.
- Multiply the measure of each element S_i by the value of f at some point x_i in the element to get an estimate of the contribution of S_i to the value of I.
- Add up all these contributions to get an estimate of the value of I.
- Determine the limit of all such sums as the measures of the elements shrink to zero. If the limit it exists, it is by definition, the exact value of I and it is called the integral of f over S.

We plan to denote a subset of $\mathbb{R}^2$, (the plane by), R and a subset of $\mathbb{R}^3$, (3-dimensional space), by B or V.

Example 1 *Let R be the rectangle with vertices at $(0,-2,0)$, $(5,-2,0)$, $(5,4,0)$, $(0,4,0)$ and f be the function of two variables with formula $f(x,y) = xy + 25$. We wish to determine the volume of the solid, (shown below), enclosed by the graph of f and the rectangle.*

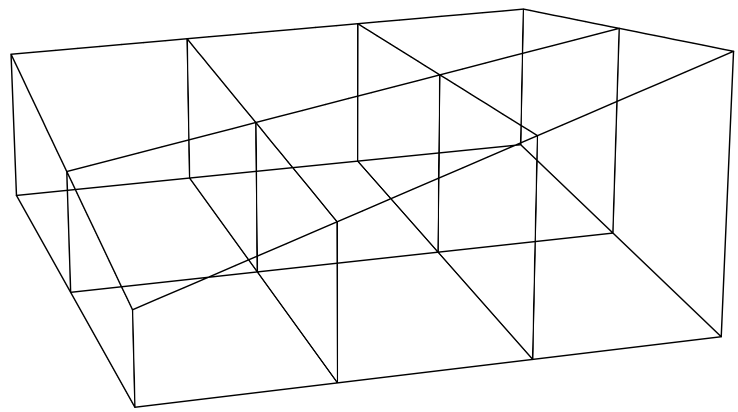


We know how to calculate the volume of any rectangular box, (simply multiply its length, width and height), but this is not one of them, therefore we have to "integrate". More precisely, we have to partition the rectangle R into smaller rectangles, use the smaller rectangles to calculate approximate values of the required volume then determine the limit of the approximations. The easiest way to partition R into smaller rectangles is to divide the intervals $[0,5]$ and $[-2,4]$ into smaller subintervals. In the figure below, we divided $[0,5]$ into two equal subintervals using the points $x_0 = 0$, $x_1 = 2.5$, $x_2 = 5$ and $[-2,4]$ into 3 equal subintervals using points $y_0 = -2$, $y_1 = 0$, $y_2 = 2$, $y_3 = 4$. They partition R into 6 smaller rectangles $R_{ij} = \{(x,y) : x_i \le x \le x_{i+1}$ and $y_j \le y \le y_{j+1}\}$, $i = 0$ or 1 and $j = 0, 1,$ or 2.



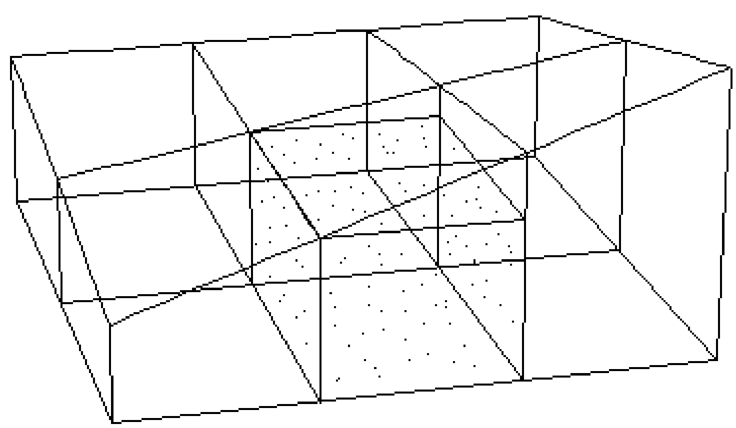
R is partitioned into smaller rectangles.

Each one of these smaller rectangles determines a small portion of the given solid, (shown below).



The smaller solids generated by the smaller rectangles

Since we know how to calculate volumes of rectangular boxes, we approximate each of the above small solids with a box that has the same base and a suitable height. To simplify the computation, we chose to approximate the small solid above the rectangle R_{ij} *by the box with base* R_{ij} *and height* $f(x_i, y_j)$. *(In general, any point* (s_i, t_j) *in* R_{ij} *may be used to get the height of an approximating box.) The box approximating the solid above* R_{11} *is shaded in the figure below. It has volume* $2.5 \times 2 \times f(x_1, y_1) = 2.5 \times 2 \times f(2.5, 0) = 125$



The total volume of the 6 approximating boxes is

$$2.5 \times 2\,[f(0,-2) + f(0,0) + f(0,2) + f(2.5,-2) + f(2.5,0) + f(2.5,2)] = 2.5 \times 2 \times 150 = 750$$

Therefore

$$\textit{Volume of solid} \;\simeq 750$$

In general, we may divide the interval $[0,5]$ *into* n *smaller subintervals* $\left[0, \frac{5}{n}\right]$, $\left[\frac{5}{n}, \frac{10}{n}\right]$, $\ldots$, $\left[\frac{5i}{n}, \frac{5(i+1)}{n}\right]$, $\ldots$, $\left[\frac{5(n-1)}{n}, 5\right]$ *of equal length* $\triangle x = \frac{5}{n}$, *(to simplify computations), and divide* $[-2,4]$ *into* m *smaller subintervals* $\left[-2, -2+\frac{6}{m}\right]$, $\left[-2+\frac{6}{m}, -2+\frac{12}{m}\right]$, $\ldots$, $\left[-2+\frac{6j}{m}, -2+\frac{6(j+1)}{m}\right]$, $\ldots$, $\left[-2+\frac{6(m-1)}{m}, 6\right]$ *of* *equal length* $\triangle y = \frac{6}{m}$*, then form the rectangles*

$$R_{ij} = \left\{ (x,y) : \tfrac{5i}{n} \leq x \leq \tfrac{5(i+1)}{n} \ \text{and} \ -2 + \tfrac{6j}{m} \leq y \leq -2 + \tfrac{6(j+1)}{m} \right\}, \ i = 0, \ldots, n-1 \ \text{and} \ j = 0, \ldots, m-1$$

They are mn *of them and each one has area* $\triangle A_{ij} = \triangle x \triangle y$*. Denote by* P_{ij} *the small solid enclosed by the graph of* f *and the rectangle* R_{ij}*. Since we know how to calculate volumes of boxes, we approximate it with a box that has the same base* R_{ij} *and height* $f(\frac{5i}{n}, -2 + \frac{6j}{m})$*. Its volume is*

$$f\left(\tfrac{5i}{n}, -2 + \tfrac{6j}{m}\right) \triangle A_{ij} = \left[\frac{5i}{n}\left(-2 + \frac{6j}{m}\right) + 25\right] \frac{5}{n} \cdot \frac{6}{m} = \left(\frac{750}{mn} - \frac{300i}{mn^2} + \frac{900ij}{m^2n^2}\right)$$

The sum of these mn *volumes is*

$$\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f\left(\tfrac{5i}{n}, -2 + \tfrac{6j}{m}\right) \triangle R_{ij} = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{750}{nm} - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{300i}{mn^2} + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{900ij}{m^2n^2}$$

Therefore the required volume is approximately equal to

$$\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{750}{nm} - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{300i}{mn^2} + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{900ij}{m^2n^2}$$

By definition, $\displaystyle\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{750}{nm} = \underbrace{\frac{750}{nm} + \frac{750}{nm} + \cdots + \frac{750}{nm}}_{nm \ terms} = 750$*. Secondly,*

$$\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{300i}{mn^2} = \left(\sum_{i=0}^{n-1} i\right)\left(\sum_{j=0}^{m-1} \frac{300}{mn^2}\right) = \left(\frac{n(n-1)}{2}\right) \frac{300m}{mn^2} = 150\left(1 - \frac{1}{n}\right).$$

In case you are puzzled, we made use of the identity $\displaystyle\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$*. Finally,*

$$\begin{aligned} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{900ij}{m^2n^2} &= \left(\frac{900}{m^2n^2} \sum_{i=0}^{n-1} i\right)\left(\sum_{j=0}^{m-1} j\right) \\ &= \frac{900}{m^2n^2}\left(\frac{n(n-1)}{2}\right)\left(\frac{m(m-1)}{2}\right) = 225\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{n}\right) \end{aligned}$$

Therefore the volume of the solid is approximately equal to

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{750}{nm} - \frac{300i}{mn^2} + \frac{900ij}{m^2n^2}\right) = 750 - 150\left(1 - \frac{1}{n}\right) + 225\left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{n}\right)$$

The limit of this sum as the widths $\triangle x_i$ *and lengths* $\triangle y_j$ *of the rectangles* R_{ij} *shrink to 0 is*

$$750 + [-150(1) + 225(1)(1)] = 825$$

This must be the volume of the solid.

We take this opportunity to introduce some notations and terminology. The sum

$$\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f\left(\tfrac{5i}{n}, -2 + \tfrac{6j}{m}\right) \triangle R_{ij} = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{750}{nm} - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{300i}{mn^2} + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \frac{900ij}{m^2n^2}$$

is called a Riemann sum of f *determined by the rectangles*

$$R_{ij} = \left\{ (x,y) : \tfrac{5i}{n} \leq x \leq \tfrac{5(i+1)}{n} \ \text{and} \ -2 + \tfrac{6j}{m} \leq y \leq -2 + \tfrac{6(j+1)}{m} \right\}, \ i = 0, \ldots, n-1 \ \text{and} \ j = 0, \ldots, m-1$$

The limit of these Riemann sums as all the $\triangle x_i$'s and $\triangle y_j$'s shrink to 0 is called the Riemann integral of f over the set R and is denoted by

$$\iint_R f(x,y)dA.$$

Why two integral signs? Because the integral is the limit of a double summation $\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{5i}{n}, -2+\frac{6j}{m}\right)\triangle A_{ij}$.

And why dA? Because we partitioned R into elements, (in this case rectangles), with ***areas*** $\triangle A_{ij}$.

Riemann Integral of a Function of two variable

We now generalize the above construction. To this end, let $f(x,y)$ be a given function of two variables and R be a set in the plane which we may assume to be a rectangle. If it is not a rectangle, simply enclose it in a suitable rectangle R and define f to have value zero outside R. Divide the rectangle into smaller rectangles R_{ij} which we may assume, for simplicity, to have the same length $\triangle x$ and the same width $\triangle y$, hence the same area $\triangle A_{ij} = \triangle x \triangle y$. Let (θ_i, α_j) be a point in the rectangle R_{ij}. The sum

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\theta_i, \alpha_j)\triangle A_{ij}. \tag{2}$$

is called a Riemann sum of f. The limit of these Riemann sums as $\triangle x$ and $\triangle y$ tend to 0, (assuming the limit exists), is called the Riemann integral of f over R and it is denoted by

$$\iint_R f(x,y)dA.$$

It may be viewed as the volume of the solid enclosed by the graph of f and the set R. As explained in Example 1, we use two integral signs because we determine the limit of a double summation. For this reason, it is often called a *double integral.* The symbol dA serves to point out that the set R was divided into elements R_{ij} with area $\triangle A_{ij}$.