Using Tangents to Determine "Good Approximations"

So far, the problems to which we have applied a derivative involved determining relative maximum or relative minimum values of given functions. We now apply it to a different type of problem; the problem of determining approximations to the values of a given function f. This is necessary in cases where approximate values are preferable to exact values because the exact values are difficult to calculate or they cannot be used to solve the problem at hand. The idea is to approximate a section of the graph of f by a tangent to its graph at an appropriate point. Since a straight line is the graph of a first degree polynomial, this process boils down to approximating f with a first degree polynomial. We then proceed to use the polynomial to solve the problem at hand.

Example 1 *To determine approximate values of* $f(x) = \sqrt{3+x}$.

Say we wish to estimate, with reasonable accuracy, the value of f *at* $x = 1.7$. *We know its exact value at the point* $x = 1$, *(it is* $f(1) = \sqrt{4} = 2$*). Furthermore,* $x = 1$ *is pretty close to the point* 1.7. *Therefore it is reasonable to consider the tangent to the graph of* f *at* $(1,2)$. *It is shown in Figure (i) below. Its slope is* $f'(1) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$, *therefore its equation is* $y - 2 = \frac{1}{4}(x-1)$ *or* $y = \frac{1}{4}(x+7)$.

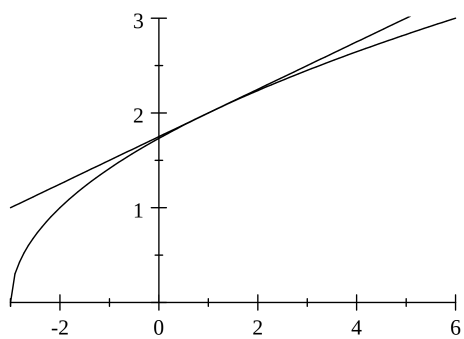


Figure (i)

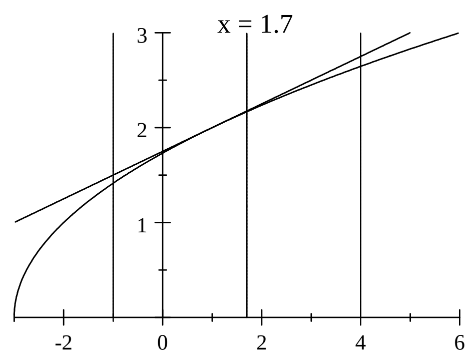


Figure (ii)

For convenience, replace y *by* $T(x)$ *so that the tangent is the graph of the first degree polynomial* T *with equation*

$$T(x) = \tfrac{1}{4}(x+7).$$

Figure (ii) shows that the graph of f *and the graph of* T *intersect the line* $x = 1.7$ *at almost the same point. This implies that* $T(1.7)$ *should be a good approximate value of* $f(1.7)$. *It is easy to calculate it and it is equal to*

$$T(1.7) = \tfrac{1}{4}(1.7+7) = \tfrac{1}{4}(8.7) = 2.175$$

Therefore we approximate $f(1.7)$ *with* $T(1.7)$ *and write*

$$f(1.7) = \sqrt{4.7} \simeq T(1.7) = \tfrac{1}{4}(1.7) + \tfrac{7}{4} = 2.175$$

A calculator gives $\sqrt{4.7} = 2.16795$, *(to 5 decimal places), which is pretty close to* 2.175. *The difference* $f(1.7) - T(1.7)$ *is called the error in the approximation.*

Figure (ii) also shows that $T(4)$ *and* $T(-1)$ *are poor approximations of* $f(4)$ *and* $f(-1)$ *respectively. In general if* x *is "far" from 1, then the point* $(x, f(x))$ *on the graph of* f *is well below the point* $(x, T(x))$ *on the tangent, hence* $T(x)$ *is a poor approximate value of* $f(x)$. *This serves to point out the general properties of approximating functions: they may be good in just a small interval.*

Exercise 2

1. *Use the tangent to the graph of* $f(x) = \sqrt[3]{x} = x^{1/3}$ *at* $(27,3)$ *to determine an approximate value of* $\sqrt[3]{25.2}$. *Use the value of* $\sqrt[3]{25.2}$ *from a calculator to estimate the error in your approximation.*

2. *Use the tangent to the graph of* $g(x) = x^{3/5}$ *at* $(32, 8)$ *to determine an approximate value of* $(33.2)^{3/5}$. *Use the value of* $(33.2)^{3/5}$ *from a calculator to estimate the error in your approximation.*

3. *Use an appropriate tangent to the graph of a suitable function to determine an approximate value of* $\dfrac{1}{\sqrt{24.3}}$. *Use the value of* $\dfrac{1}{\sqrt{24.3}}$ *from a calculator to estimate the error in your approximation.*

4. *Let* $f(x) = (1+x)^n$. *Use the tangent to the graph of* f *at* $x = 0$ *to show that when* x *is close to 0 then* $(1+x)^n \simeq 1 + nx$. *Use this to estimate (a)* $\sqrt[4]{1.036}$ *and (b)* $(1.004)^{45}$

Generalizing

We now generalize what we have done above. Thus let f be a differentiable function and c be a point in its domain. We wish to use the points on the tangent at $(c, f(c))$ to estimate the values of f at points x close to c. The first step is to determine the equation of the tangent. Its slope is $f'(c)$, therefore its equation is given by $y - f(c) = f'(c)(x - c)$, which may be written as

$$y = f'(c)(x - c) + f(c).$$

As we did above, denote y by $T(x)$ then write

$$T(x) = f'(c)(x - c) + f(c). \tag{1}$$

The tangent line at $(c, f(c))$ lies flat on the graph of f at $(c, f(c))$. This suggests that when x is close to c, then the point $(x, f(x))$ on the graph of f is close to the point $(x, T(x))$ on the tangent as shown below.

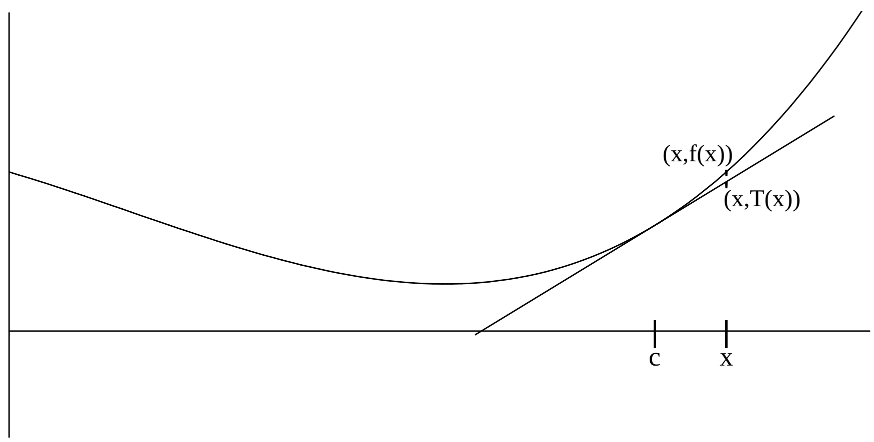


Figure (iii)

Therefore $T(x) = f'(c)(x - c) + f(c)$ should be a good approximate value of $f(x)$. More precisely, when x is close to c then

$$f(x) \simeq f'(c)(x - c) + f(c) \tag{2}$$

In some instances it is necessary to subtract $f(c)$ from both sides to get

$$f(x) - f(c) \simeq f'(c)(x - c) \tag{3}$$

The number $x-c$ is called a change in the independent variable x and it is denoted by $\triangle x$ which is pronounced "delta x". Thus

$$\triangle x = x - c$$

Likewise, the number $f(x) - f(c)$ is called the change in f corresponding to the change in x of $\triangle x$ units and it is denoted by $\triangle f$, which is pronounced "delta f". Since $x = c + \triangle x$, we may write $f(x)$ as $f(c + \triangle x)$. Therefore

$$\triangle f = f(x) - f(c) = f(c + \triangle x) - f(c)$$

and (3) may be written as

$$\triangle f \simeq f'(c)\triangle x$$

Some common terms:

- $\triangle f = f(x) - f(c)$ *is called the actual change in the value of* f
- $f'(c)\triangle x$ *is called the estimated change in the value of* f
- *The fraction* $\dfrac{\triangle f}{f(c)}$ *is called the* ***actual fractional change*** *in the value of* f *at* c*.*
- *The percentage* $\dfrac{\triangle f}{f(c)} \times 100$ *is called the* ***actual percentage change*** *in the value of* f *at* c*.*
- *The fraction* $\dfrac{f'(c)\triangle x}{f(c)}$ *is called the* ***estimated fractional change*** *in the value of* f *at* c*.*
- *The percentage* $\dfrac{f'(c)\triangle x}{f(c)} \times 100$ *is called the* ***estimated percentage change*** *in the value of* f *at* c*.*

As you will see in several examples to follow, the estimated change is used to approximate the exact change.

Example 3 *When a sphere of radius 10 cm was heated, its radius increased to 10.03 cm. Recall that the volume of a sphere with radius* r *is* $V(r) = \frac{4}{3}\pi r^3$*. Therefore the volume of the sphere before being heated was* $V(10) = \frac{4}{3}(1000\pi)$ *cu. cm. The change in its radius was* $\triangle r = 0.03$ *cm, hence the actual change in volume is*

$$\triangle V = V(10.03) - V(10) = 12.036036\pi.\ cu.\ cm$$

Since $v'(r) = 4\pi r^2$*, the estimated change in volume is*

$$V'(10)\triangle r = 4\pi(10)^2(0.03) = 12\pi.\ cu.\ cm$$

The actual fraction change in volume is

$$\frac{12.036036\pi \times 3}{4\pi(10)^3} = 0.00903\ cu.\ cm$$

The estimated fractional change is

$$\frac{4\pi(10)^2(0.03) \times 3}{4\pi(10)^3} = 0.009\ cu.\ cm$$

The actual percentage change in volume is

$$\frac{12.036036\pi \times 3}{4\pi(10)^3} \times 100\% = 0.903\%,$$

The estimated percentage change in volume is

$$\frac{12\pi \times 3}{4\pi(10)^3} \times 100\% = 0.9\%.$$

Example 4 *It is required to draw a circle of radius* 5 *cm. Inevitably, there is an error in the measurement of its radius. What level of accuracy must we seek so that the error in the area of the circle does not exceed 1 square centimeter?*

***Solution**: The area of a circle of radius r is $A(r) = \pi r^2$. If the error in measuring the radius is $\triangle r$ then we draw a circle of radius $(5 + \triangle r)$, instead of a circle of radius 5. The actual error in the area is*

$$\triangle A = \pi (5 + \triangle r)^2 - 25\pi = \pi \left(10\triangle r + (\triangle r)^2\right)$$

We need $\triangle r$ such that

$$\left|\pi \left(10\triangle r + (\triangle r)^2\right)\right| < 1$$

This is a quadratic inequality which is relatively hard to solve. To solve a simpler one, we determine $\triangle r$ such that the estimated error is less than 1 sq. cm. Thus we look for $\triangle r$ such that

$$|V'(5)\triangle r| = |2\pi(5)\triangle r| = |10\pi\triangle r| < 1$$

This is a linear inequality and we easily solve it to get $|\triangle r| < \frac{1}{10\pi} = 0.032$. Therefore we should use an instrument that can measure r accurately to about 0.03 cm.

Differentials

Let f be a given function with a derivative $f'(c)$ at a point c. Let $T(x) = f(c) + f'(c)(x - c)$ be the first degree polynomial we used to approximate the values of f in some vicinity of c. Thus when x is close to c then

$$f(x) \simeq f(c) + f'(c)(x - c) \tag{4}$$

It is convenient to convert (4) into an equation by introducing the error $e(x) = f(x) - T(x)$ in the approximation. Then we may write

$$f(x) = T(x) + e(x) = f(c) + f'(c)(x - c) + e(x)$$

or

$$f(x) - f(c) = f'(c)(x - c) + e(x) \tag{5}$$

Using the notation $x = c + \triangle x$ which we introduced above, (5) may be written as

$$f(c + \triangle x) - f(c) = f'(c)\triangle x + e(x) \tag{6}$$

The expression $f'(c)\triangle x$ is linear in the variable $\triangle x$. We use it to introduce the **differential of f at c**, denoted by df_c. It is the function with formula

$$df_c(\triangle x) = f'(c)\triangle x$$

The notation df_c emphasizes that the differential depends on the point c. If you choose another point b in the domain of f, you get a differential df_b at b which may be different from df_c.

Using differentials, (6) may be written as

$$\triangle f = df_c(\triangle x) + e(x)$$

When $\triangle x$ is small, the error term $e(x)$ is small and so $\triangle f \simeq df_c(\triangle x)$. In other words, when the independent variable changes by $\triangle x$, then the corresponding change in f is approximately equal to the value of df_c at $\triangle x$.

Example 5 *Let $f(x) = \sin 2x$, $c = \frac{\pi}{3}$ and $b = \frac{3\pi}{4}$.*

1. *Then $f'(c) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, therefore the differential of f at c is df_c with formula*

$$df_c(\triangle x) = \frac{1}{2}\triangle x$$

When, for example, x changes by -0.1 from $\frac{\pi}{3}$ to $\frac{\pi}{3} - 0.1$ then the value of f changes by approximately -0.05 from $\frac{1}{2}$ to $\frac{1}{2} - 0.05 = 0.45$.

2. *Since* $f'(b) = \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, *the differential of* f *at* b *is* df_b *with formula*

$$df_b(\triangle x) = -\frac{1}{\sqrt{2}}\triangle x$$

Exercise 6 *When required, the volume of a sphere with radius* r *is* $\frac{4}{3}\pi r^3$ *and the volume of a box with length* ℓ*, width* w *and height* h *is* ℓwh.

1. *Give a formula for the differential of:*

$$\text{(a) } f(x) = x^3 + x \text{ at } c = 1 \qquad \text{(b) } g(x) = x\sqrt{5 - x^2} \text{ at } c = 2.$$

2. *A spherical ornament has a radius of 1 cm. It is to be given a silver coating that is* 0.03 *cm thick. Show that the volume of paint used is* $\frac{4}{3}\pi\,(2.03)^3 - \frac{4}{3}\pi\,(2)^3$ *then use a differential to estimate it.*

3. *You have a cubical metallic block, (i.e. a block whose length, width and height are all equal), whose volume has to be determined. You measure its edge and find it to be* 6 ± 0.2 *cm. Say you compute the volume of the cube using 6 as the length of each edge. Use a differential to estimate the error in the volume you get. Also compute the estimated percentage error in the volume.*

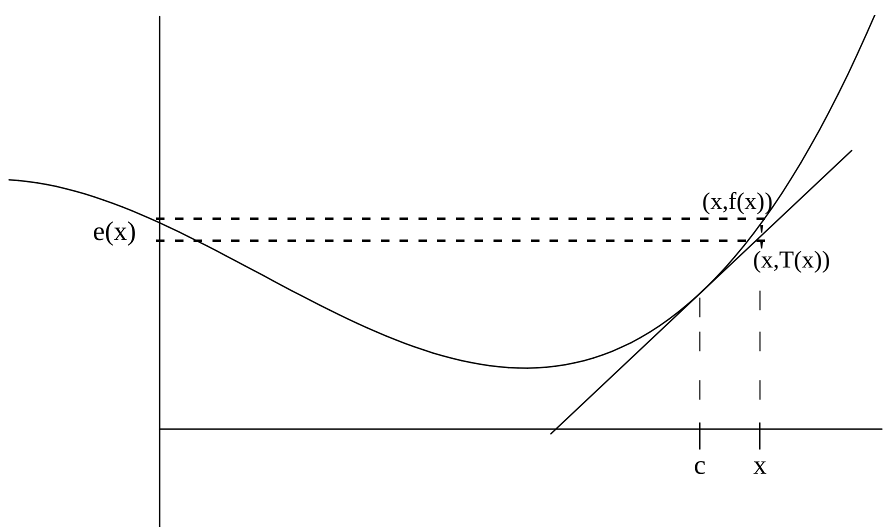
4. *The surface area of a right circular cone with base radius* r *and height* h *is* $A = \pi r\sqrt{r^2 + h^2}$. *Use a differential to estimate the change in area when the radius changes from 8 cm to 8.08 cm while the height remains unchanged and equal to 12 cm.*

5. *It is required to measure the radius of a sphere then calculate its volume with an error of no more than 1% of its true value. Determine the largest estimated percentage error that can be tolerated in the measurement of* r.

6. *You are required to construct a box with a square base and a height equal to one third the length of a side of its base. Its volume must be 72 cubic feet. Use a differential to estimate how accurately the length of its base should be made so that the volume of the box is accurate to within* 0.9 *cubic feet.*

Another View of a Derivative

As before, let a given function f have derivative $f'(c)$ at a point c, let $T(x) = f(c) + (x - c)f'(c))$ be the first degree polynomial that approximates f in the vicinity of c and $e(x) = f(x) - T(x)$ be the error in the approximation at a point x close to c. In the figure below $e(x)$ is the distance between the two dotted



horizontal lines. We want to compare this to the distance between the two dotted vertical lines, which happens to be $|x - c|$, the distance between x and c. Because the graph of f lies flat on the tangent line,

$e(x)$ is must be much smaller than $|x-c|$, (as long as x is close to c). In fact, for a differentiable function f, $e(x)$ is so much smaller than $|x-c|$ that the quotient $\dfrac{e(x)}{|x-c|}$ approaches 0 as x approaches c.

For a specific example, take $f(x)=x^3$ *and* $c=1$. *Then*

$$T(x)=f(1)+(x-1)\,f'(1)=3\,(x-1)+1$$

Therefore the error at a point x *close to* 1 *is*

$$\begin{aligned} e(x) &= f(x)-T(x)=x^3-3\,(x-1)-1 \\ &= \left(x^3-1\right)-3\,(x-1)=(x-1)\left(x^2+x+1\right)-3\,(x-1) \\ &= (x-1)^2\,(x+2) \end{aligned}$$

Take a point like $x=1.2$ *close to* $c=1$. *Its distance from* c *is* 0.2 *and the error* $e(1.2)$ *is* $(0.2)^2\,(1.2)=0.048$ *which is much smaller than* 0.2. *Furthermore,*

$$\frac{e(x)}{(x-1)}=(x-1)\,(x+2)$$

which approaches 0 *as* x *approaches* 1.

The above observation is used to give the following view of a derivative:

Definition 7 *The derivative of a function* f *at a point* c *is the number, denoted by* $f'(c)$, *with the property that the quotient*

$$\frac{f(x)-f(c)-(x-c)\,f'(c)}{x-c}$$

approaches 0 *as* x *approaches* c.

We may replace this definition by another one that does not involve quotients as follows:

Let $b(x)=\dfrac{f(x)-f(c)-(x-c)\,f'(c)}{x-c}$. Then the values of $b(x)$ approach 0 as x approaches c. Now re-arrange to get

$$f(x)=f(c)+(x-c)\,f'(c)+(x-c)\,b(x)$$

This suggests the following definition:

Definition 8 *The derivative of a function* f *at a point* c *is the number, denoted by* $f'(c)$, *with the property that*

$$f(x)=f(c)+(x-c)\,f'(c)+(x-c)\,b(x)$$

where b *is a function whose values approach* 0 *as* x *approaches* c.

We will use this form in proving some statements later on.