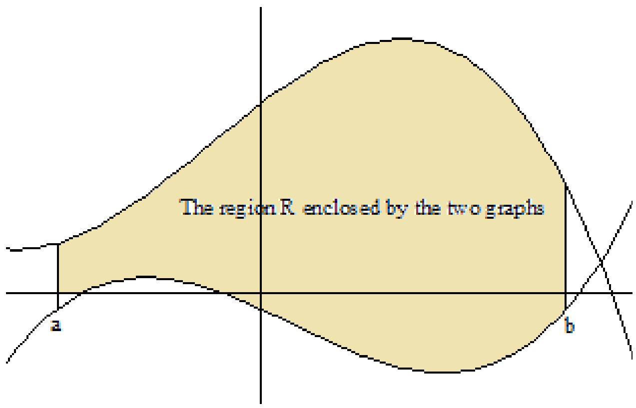
Some Applications of Antiderivatives

The applications of antiderivatives we now consider boil down to approximating some physical quantity with Riemann sums. We then define the exact value of the quantity to be the limit of the approximating Riemann sums. The limit is evaluated using an antiderivative of some function.

Area Enclosed By Two Curves

The problem of determining the area of the region enclosed by two curves is a generalization of the problem, we have already encountered, of determining the area of the region between the graph of a given function and a section of the horizontal axis. We now replace the horizontal axis with a more general curve.

Let f and g be given functions with the property that the graph of f is above the graph of g on some given interval $[a, b]$. We wish to calculate the area of the region R enclosed by the two curves and the two vertical lines $x = a$ and $x = b$.



We do what we have done many times before, which is:

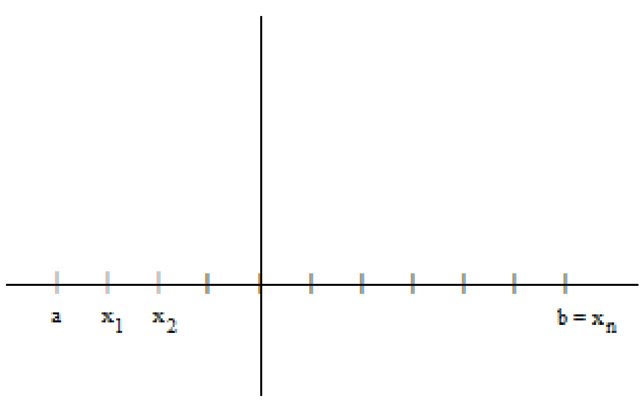
- Calculate approximate values of the area. We called them Riemann sums.
- Determine the limit of the Riemann sums.

As we have pointed out before, the limit of the Riemann sums is the best candidate for the exact value of the area.

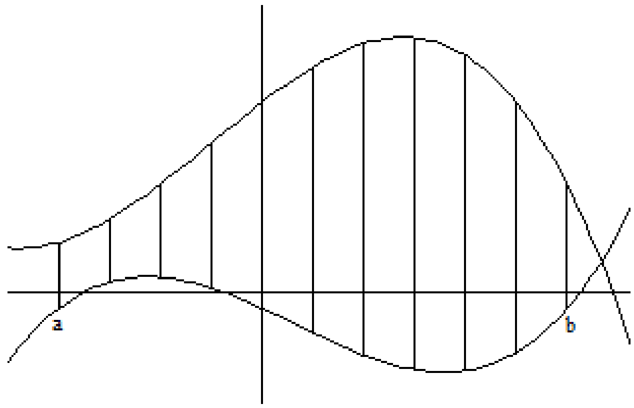
Calculating approximate values of the area is pretty standard. We do the following:

1. Divide the interval $[a, b]$ into small subintervals. For convenience, we divide it into n equal subintervals $[a, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$. Each one has length

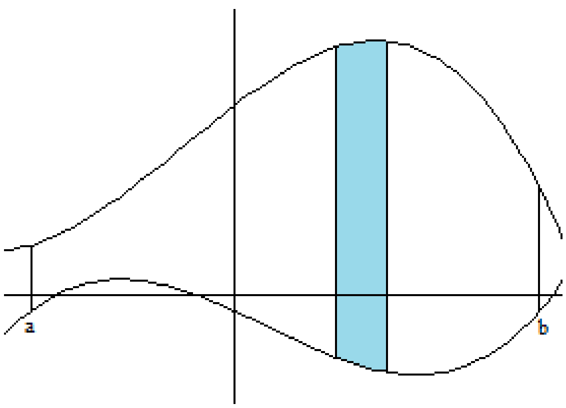
$$\triangle x = \frac{b-a}{n}$$



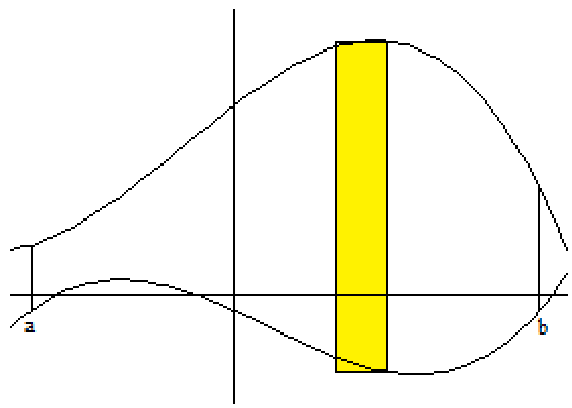
2. Consider the strips of the region determined by each subinterval.



3. Approximate each strip by a rectangle of width $\triangle x$ and a suitable height. A typical one determined by an interval $[x_i, x_{i+1}]$ is shown below. It is approximated by a rectangle with width $\triangle x$ and height $[f(x_i) - g(x_i)]$. Its area is $[f(x_i) - g(x_i)]\,\triangle x$.



A typical strip



An approximating rectangle

4. Determine the total area of the n approximating rectangles. This is the Riemann sum determines by the n subintervals and points x_1, x_2, $\ldots$, x_n. It is equal to

$$[f(x_1) - g(x_1)] \cdot \triangle x + [f(x_2) - g(x_2)] \cdot \triangle x + \cdots + [f(x_n) - g(x_n)] \cdot \triangle x = \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\triangle x$$

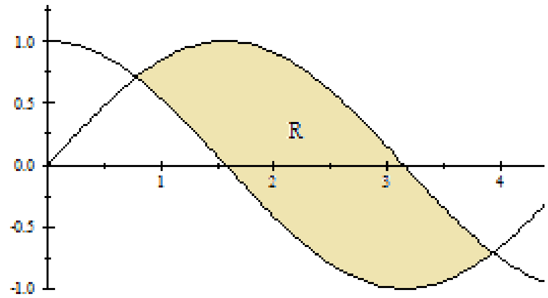
The limit of such sums as $\triangle x \to 0$ is

$$\int_a^b [f(x) - g(x)]\,dx$$

and it is defined to be the area enclosed by the two curves.

Example 1 *To calculate the area of R, (shown below), enclosed by the graphs of $f(x) = \sin x$ and $g(x) =$*

$\cos x$ *on the interval* $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.



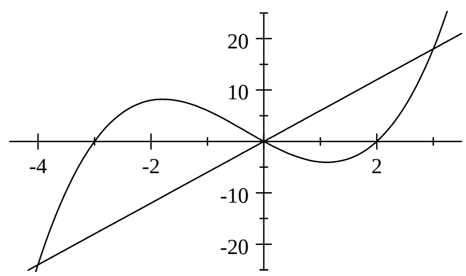
According to the above result the required area is

$$\int_{\pi/4}^{5\pi/4} [f(x) - g(x)]\, dx = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx = \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$

$$= -\left[\left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}\right) - \left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\right)\right] = 2\sqrt{2}$$

Example 2 *To determine the area of the region enclosed by the graphs of* $f(x) = 6x$ *and* $g(x) = x\,(x-2)\,(x+3)$.

The two graphs are shown below.



They intersect at points (x, y) *where* x *satisfies the equation*

$$6x = x\,(x-2)\,(x+3)\,.$$

It has solution $x = -4$, *or* 0 *or* 3. *To determine the required area, note that the graph of* g *is above the graph of* f *on the interval* $(-4, 0)$, *but it is below the straight line on the interval* $(0, 3)$. *Therefore the area they enclose is*

$$\int_{-4}^{0} [g(x) - f(x)]\, dx + \int_{0}^{3} [f(x) - g(x)]\, dx$$

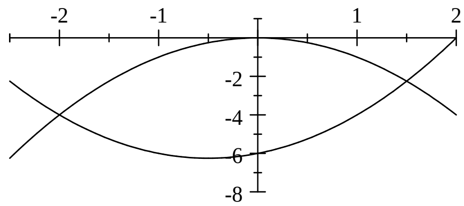
Since $f(x) - g(x) = 12x - x^3 - x^2$ *and* $g(x) - f(x) = x^3 + x^2 - 12x$, *the area is*

$$\int_{-4}^{0} \left(x^3 + x^2 - 12x\right) dx + \int_{0}^{3} \left(12x - x^3 - x^2\right) dx$$

Evaluate this. You should get the answer $78\frac{1}{12}$ *square units.*

Exercise 3

1. *The figure below shows the graphs of* $f(x) = -x^2$ *and* $g(x) = x^2 + x - 6$.

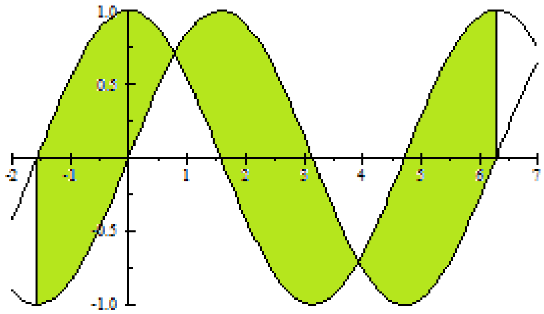


   (a) *Determine the coordinates of the points where the two graphs intersect.*

   (b) *Let* R *be the region enclosed by the two graphs. Imagine dividing it into* n *strips. Draw a typical one and a rectangle that approximates it.*

   (c) *Calculate the area of* R.

2. *Calculate the area of* R, *(shown below), enclosed by the graphs of* $f(x) = \sin x$ *and* $g(x) = \cos x$ *on the interval* $\left[-\frac{1}{2}\pi, 2\pi\right]$.



3. *Calculate the area of the region enclosed by:*

   (a) *The graphs of* $f(x) = x^3$ *and* $g(x) = -x^3$ *on the interval* $[-1, 2]$.

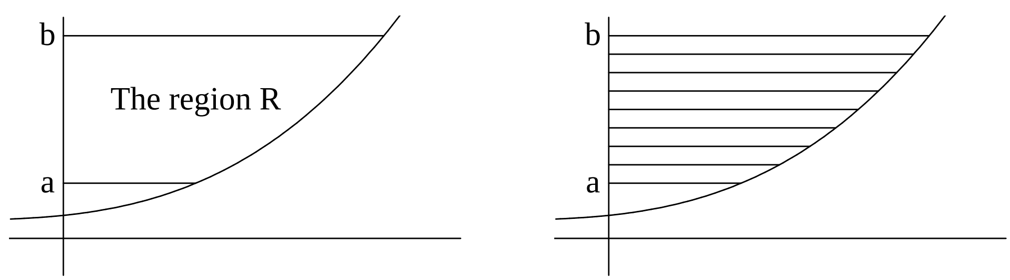
   (b) *The* x*-axis and the graph of* $f(x) = xe^x$ *on the interval* $[-3, 0]$. *(First sketch the graph.)*

   (c) *The* x *axis and the graph of* $f(x) = x^3 - x^2 - 2x$. *(First sketch the graph.)*

   (d) *The graph of* $f(x) = x^2 + 2x + 3$, *(a parabola), and the graph of* $g(x) = x + 9$, *(a straight line).*

   (e) *The graph of* $f(x) = x^2 + 5x + 1$, *(a parabola), and the graph of* $g(x) = -x^2 + 2x + 3$, *(also a parabola).*

4. *Let* f *be a function with an inverse* f^{-1}. *Consider the region* R *enclosed by the graph of* f, *the* y*-axis and the two lines* $y = a$ *and* $y = b$. *You may assume that* R *is in the first quadrant as depicted in the figure below. Subdivide it into smaller strips as shown in the second figure.*



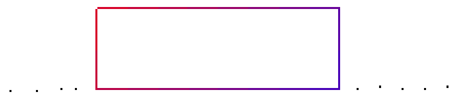
Approximate each strip with a rectangle then calculate an approximate value of the area of R.

   (a) *Use the approximation to deduce that the area of* R *is* $\displaystyle\int_a^b f^{-1}(y)dy$.

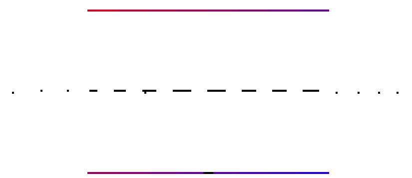
   (b) *Use the formula in part (a) to calculate the area of the region* R *enclosed by the graph of* $f(x) = 2x^3$, *the* y*-axis and the two lines* $y = 0$ *and* $y = 16$.

Volumes of Solids of Revolution by the Disc and Shell Methods

One gets a solid of revolution when one revolves a given region in the plane, about some given axis, through 4 right angles. The simplest one is a cylinder obtained by revolving, (about the x-axis), a rectangle with one edge on the x-axis. To sketch it, draw the reflection of the line segment in the x-axis as shown.



Rectangle to be rotated



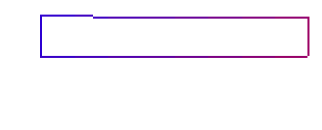
Line segment and its reflection

Now join the ends with curves as shown below. To give the cylinder a more realistic look, dot what should be the invisible part of the curve at the back.

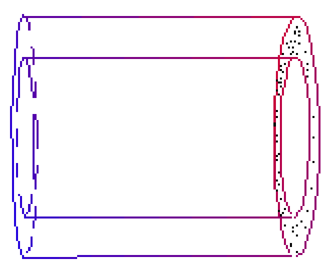
Join both ends with curves

The invisible edge is dotted

If you rotate, (about the x-axis), a rectangle that is above the x-axis, you get a solid called a **shell**. (It is the solid between two cylinders with the same axis.)

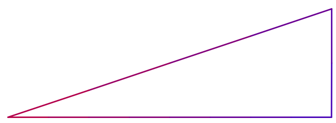


A rectangle above the x-axis

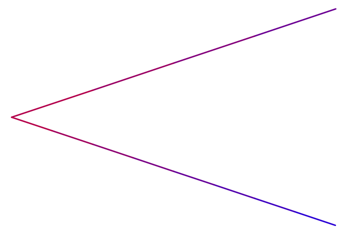


A shell

The next simple one to draw is a right circular cone obtained by revolving the region enclosed by a right triangle about one of its sides, (not the hypotenuse). The procedure for drawing it is similar; start by drawing a reflection of the hypotenuse.

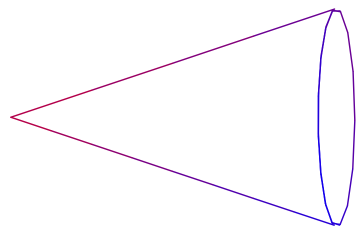


Region to be rotated



Hypotenuse and its reflection

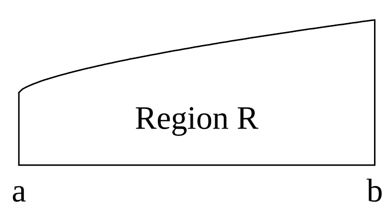
Now join the two ends of the line segments with a curve as shown below.

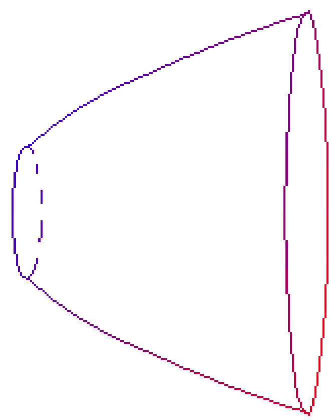


A right circular cone

The Disc Method

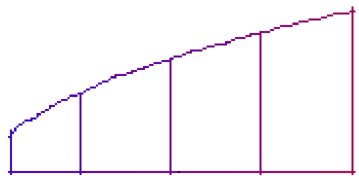
Consider the region R in the plane enclosed by the x-axis and the graph of some function f on an interval $[a, b]$. Imagine revolving it about the x-axis through 4 right angles to get a solid of revolution.



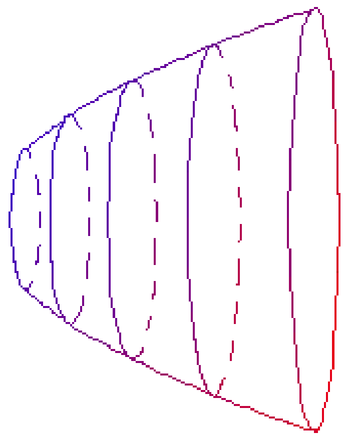


We wish to determine its volume. Let it be V. We may determine it as follows:

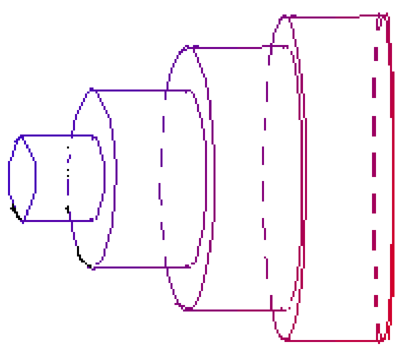
Divide $[a, b]$ into smaller subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$ and use the subintervals to partition R into smaller strips. (In the figure below, $[a, b]$ is divided into 4 subintervals.)



Note that each strip is perpendicular to the axis of rotation. When the graph is rotated about the x-axis, each strip determines a slice of the solid as shown in the figure below.



The slice determined by the interval $[x_i, x_{i+1}]$ is approximated with a disc of radius $f(x_i)$, thickness $\triangle x = (x_{i+1} - x_i)$ and volume $\pi\,[f(x_i)]^2\,\triangle x$, (see the figure below).



The sum of the volumes of the approximating discs is an approximate value of V. In other words,

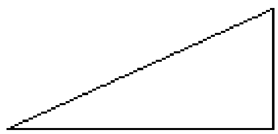
$$V \simeq \pi \sum_{i=1}^{n} [f(x_i)]^2 \triangle x$$

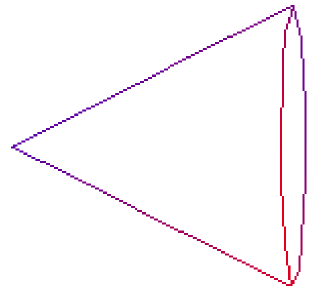
The exact value of V is the limit of the Riemann sums $\quad \pi \sum_{i=1}^{n} [f(x_i)]^2 \triangle x \quad$ as $\triangle x \to 0$. That limit is

$$V = \pi \int_a^b [f(x)]^2\, dx \tag{1}$$

Because we obtained formula (1) by approximating the solid with appropriate discs, this method of computing volumes is called the "**disc method**".

Example 4 *Take $f(x) = \frac{1}{2}x$, and let R be the triangle, shown below, enclosed by the graph of f on the interval $[0, 2]$ and the x-axis. Consider the right circular cone obtained by revolving R through 4 right angles about the x-axis.*





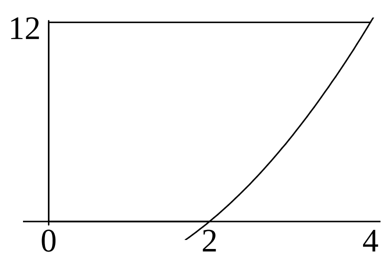
Its volume is

$$V = \int_0^2 \pi \left(\frac{1}{2}x\right)^2 dx = \pi \int_0^2 \frac{1}{4}x^2 dx = \frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^2 = \frac{2\pi}{3}.$$

You probably know that the volume of a right circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$. Use this formula to double check the above result.

Exercise 5

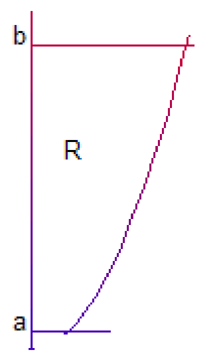
1. *Let R be the region enclosed by the graph of $f(x) = 2 + x^2$, $0 \leq x \leq 2$, (a parabola), and the x-axis. Show that the volume of the solid generated by revolving R about the x-axis through 4 right angles is $\frac{376\pi}{15}$ square units.*
2. *Let R be the region enclosed by the graph of $g(x) = \sin x$, $0 \leq x \leq \pi$ and the x-axis. It is revolved about the x-axis through 4 right angles. Show that the volume of the solid generated is $\frac{1}{2}\pi^2$ square units.*
3. *Consider the function $f(x) = \dfrac{x}{1 + x^2}$.*
   - (a) *Sketch the graph of f.*
   - (b) *Let R be the region enclosed by the x-axis and the graph of f between $x = \frac{1}{4}$ and $x = 4$. Show that its area is $\ln 2$ square units.*
   - (c) *A wine bottle is obtained by revolving the region R about the x-axis through 4 right angles. Its volume, in cubic units, must be*
     $$\pi \int_{1/4}^4 \frac{x^2 dx}{(1 + x^2)^2}.$$
     Show that $\displaystyle\int \frac{x^2 dx}{(1 + x^2)^2} = \frac{1}{2}\left(\arctan x - \frac{x}{1 + x^2}\right) + c$, then verify that the volume of the bottle is $\frac{\pi}{2}\left(\tan^{-1} 4 - \tan^{-1} \frac{1}{4}\right)$ cubic units.
4. *The figure shows the graph of $f(x) = x^2 - 4$, $2 \leq x \leq 4$ and the three line segments; one of them joining $(4, 12)$ to $(0, 12)$, another one joining $(0, 12)$ to $(0, 0)$ and the third one joining $(0, 0)$ to $(2, 0)$. The region they enclose is*

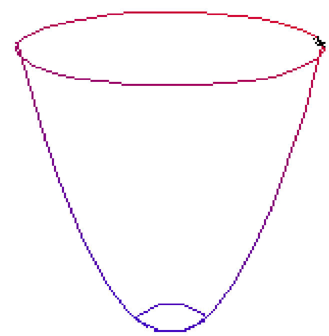


rotated about the x-axis through 4 right angles. Explain why the volume of the solid generated must be $\pi \int_0^4 144dx - \pi \int_2^4 \left(x^2 - 4\right)^2 dx$ *units then evaluate the integrals.*

5. *Let* $f(x) = x^2$ *and* $b > 0$*. Consider the region enclosed by the x-axis and the graph of f on the interval* $[0, b]$*. It is rotated about the x-axis through 4 right angles. For what value of b is the volume of the solid generated equal to 2 cubic units?*

6. *Let f be a function with an inverse* f^{-1}*. Suppose the region R enclosed by the graph of f, the y-axis and the two lines* $y = a$ *and* $y = b$*, is in the first quadrant as shown in the figure below.*





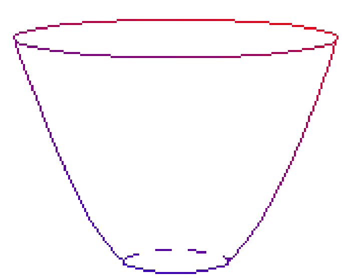
Solid of revolution

   (a) *Show that the volume of the solid obtained by revolving R about the y-axis through 4 right angles is* $\pi \int_a^b \left[f^{-1}(y)\right]^2 dy$*.*

   (b) *In particular, let R be the region enclosed by the graph of* $f(x) = x^3$*, the y-axis, the line* $y = 1$ *and the line* $y = 8$*. Use the result of part (a) to calculate the volume of the solid obtained by revolving R about the y axis through 4 right angles.*

7. *Let* $h > 0$ *and R be the region enclosed by the x-axis and the graph of* $f(x) = \sqrt{1 + x}$ *on* $[0, h]$*.*

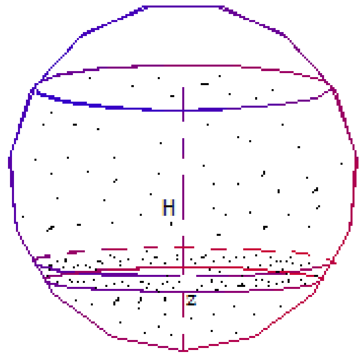
   (a) *A container is obtained by revolving the graph of* $f(x) = \sqrt{1 + x}$*, on the interval* $[0, h]$*, through 4 right angles about the x-axis. Rotate it counter-clockwise so that it is upright as shown in the figure below.*



   Assume that it is closed at the bottom but open at the top. Show that when it is full, it contains $\pi \left(h + \frac{1}{2}h^2\right)$ *cubic units of fluid.*

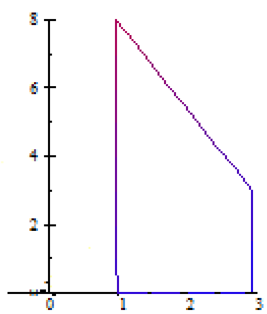
   (b) *Assume that x and f(x) are measured in meters. Water is poured into the container at the rate of* $\frac{1}{2}$ *cubic meters per minute. At what rate is the water level rising when it is 1 meter deep?.*

8. *A spherical water tank has radius R feet. Show that when the water level is H feet deep, then the volume of water in the tank is* $\pi\left(RH^2 - \frac{H^3}{3}\right)$ *cubic feet. (Imagine dividing the water into thin layers. A typical layer z feet from the bottom of the tank is shown in the figure below. Show that it can be approximated by a thin disc with radius* $\sqrt{R^2-(R-z)^2}$ *feet. Estimate its volume and form a Riemann sum.)*

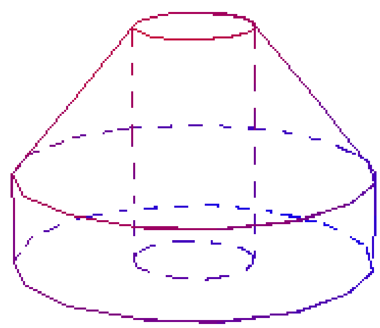


The Method of Shells

To introduce the method of shells, consider the region R enclosed by the graph of $f(x) = -\frac{5}{2}x + \frac{21}{2}$ in the interval $[1,3]$ and the x-axis. Imagine revolving it about the y-axis. The result is the solid of revolution sketched below.

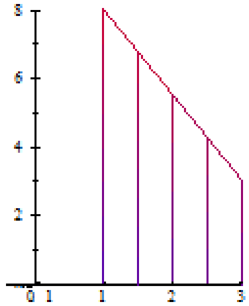


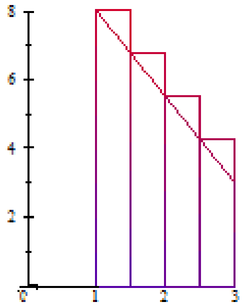
The region R



The solid of revolution

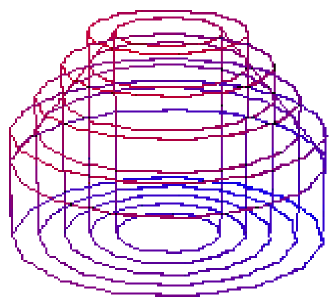
To calculate its volume, partition the interval $[1,3]$ into n smaller subintervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$ of length $\triangle x = \frac{2}{n}$ each. This partitions R into n smaller strips. (In the left figure below, R is partitioned into 4 smaller strips.) Note that, this time, each strip is *parallel* to the axis of rotation. Approximate each strip with a suitable rectangle as shown in the right figure.





R partitioned into smaller strips   Rectangles that approximate the strips

Now revolve each rectangle about the y-axis. The result are n shells whose total volume approximates the volume of the solid. The figure below shows the four shells generated by the above rectangles. The solid of revolution can be seen inside the shells.



The shell generated by the rectangle on a typical interval $[x_i, x_{i+1}]$ has volume

$$\pi (x_{i+1})^2 f(x_i) - \pi (x_i)^2 f(x_i) = \pi (x_{i+1} + x_i)(x_{i+1} - x_i) f(x_i)$$

Since $x_{i+1} = x_i + \triangle x$, it follows that the volume of the shell is $\pi (2x_i + \triangle x) f(x_i)\triangle x$, which is approximately equal to $2\pi x_i f(x_i)\triangle x$, because $2x_i + \triangle x \simeq 2x_i$ when $\triangle x$ is close to 0. Therefore the volume of the solid is approximately equal to

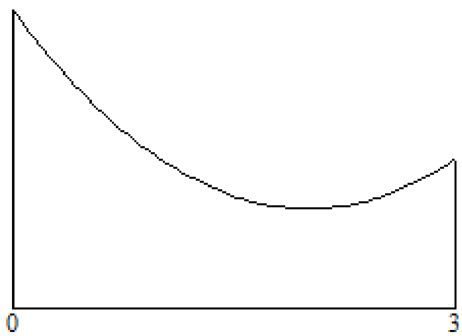
$$\sum_{i=1}^{n} 2\pi x_i [f(x_i)] \triangle x = 2\pi \sum_{i=1}^{n} x_i [f(x_i)] \triangle x$$

The exact volume is the limit as $\triangle x \to 0$ of the above Riemann sums, which is

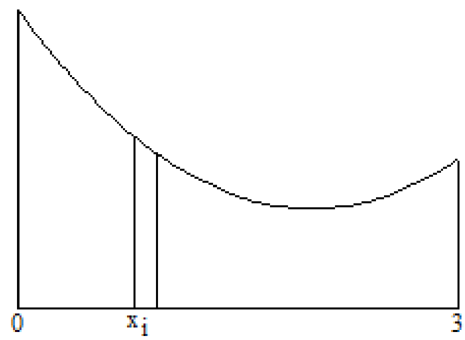
$$2\pi \int_1^3 x f(x) dx = 2\pi \int_1^3 \left(-\frac{5x^2}{2} + \frac{21x}{2} \right) dx = 2\pi \left[-\frac{5x^3}{6} + \frac{21x^2}{4} \right]_1^3 = \frac{122\pi}{3}$$

In general, let R be a given region in the plane and V be the volume of the solid generated when R is revolved through 4 right angles about a given axis. To calculate V using the method of shells, (a) partition R into smaller strips **parallel** to the axis of rotation, (b) calculate the volume of the shell generated by each strip and sum up, (c) take limits.

Example 6 *Let R be the region enclosed by the graph of $f(x) = x^2 - 4x + 6$ on the interval $[0, 3]$ and the x-axis. Imagine revolving it about the y-axis through 4 right angles to generate a solid. Let the volume of* *the solid be V. To calculate it using the method of shells, divide $[0,3]$ into smaller subintervals*

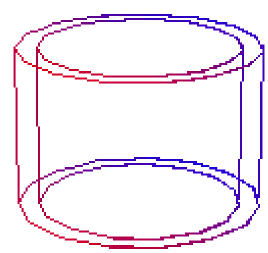


The region R



A typical strip

$[x_0,x_1]$, $[x_1,x_2]$, $\ldots$, $[x_{n-1},x_n]$. *These partition R into smaller strips parallel to the y-axis, (which is the axis of rotation). A typical one on some interval $[x_i,x_{i+1}]$ is shown above. Approximate it with a rectangle whose base is the interval $[x_i,x_{i+1}]$, and whose height is $f(x_i)$. The rectangle generates a shell with inner radius x_i and height $f(x_i)$ shown below.*



The shell with inner radius x_i and height $f(x_i)$

Its volume is approximately equal to

$$2\pi x_i\left[f(x_i)\right]\triangle x = 2\pi x_i\left[x_i^2-2x_i+2\right]\triangle x.$$

Therefore V is approximately equal to the Riemann sum

$$2\pi\sum_{i=1}^{n} x_i\left[x_i^3-2x_i^2+2x_i\right]\triangle x.$$

The exact value of V is the limit of such Riemann sums as $\triangle x\to 0$, which is

$$2\pi\int_0^3\left(x^3-2x^2+2x\right)dx = 2\pi\left[\frac{x^4}{4}-\frac{2x^3}{3}+x^2\right]_0^3 = \frac{45\pi}{2}$$

In the next example, the region is revolved about the x-axis

Example 7 *Let R be the region enclosed by the lines $y_1=\frac{1}{3}x+1$, $y_2=2x-4$ and the x-axis, (see the figure in Example **??** on page **??**). We calculate the volume of the solid generated by revolving R about the x-axis through 4 right angles. The region lies between $y=0$ and $y=2$. Partition the interval $[0,2]$ into n subintervals. This partitions R into n strips. A typical strip determined by an interval $[y_i,y_{i+1}]$ generates a shell with inner radius radius y_i, approximate length $\frac{1}{2}(y_i+4)-3(y_i-1)$ and thickness $\triangle y=(y_{i+1}-y_i)$. Its volume is approximately equal to $2\pi y_i\left[\frac{1}{2}(y_i+4)-3(y_i-1)\right]\triangle y$. Therefore the volume of the solid is approximately*

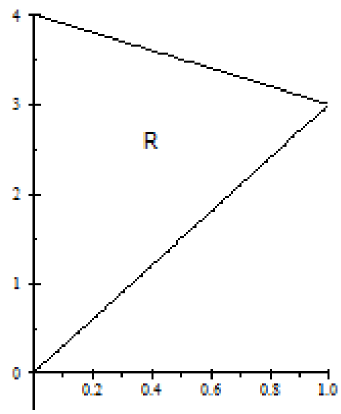
$$2\pi\sum_{i=0}^{n-1} y_i\left[\frac{1}{2}(y_i+4)-3(y_i-1)\right]\triangle y = 2\pi\sum_{i=0}^{n-1} y_i\left(5-\tfrac{5}{2}y_i\right)\triangle y$$

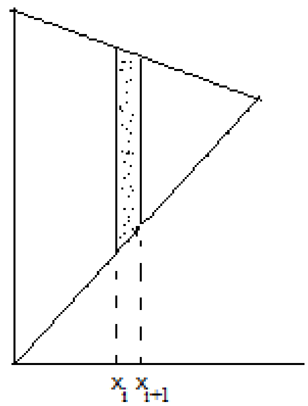
Its exact volume is the limit of the above Riemann sums as $\triangle y \to 0$, which is

$$2\pi \int_0^2 y\left(5 - \tfrac{5}{2}y\right) dy = 10\pi \int_0^2 \left(y - \tfrac{1}{2}y^2\right) dy = 10\pi \left[\frac{y^2}{2} - \frac{y^3}{6}\right]_0^2 = \frac{20\pi}{3}$$

Exercise 8

1. *In the left figure below, R is the region enclosed by the y-axis, the line $y = 3x$ and the line $y = 4 - x$.*





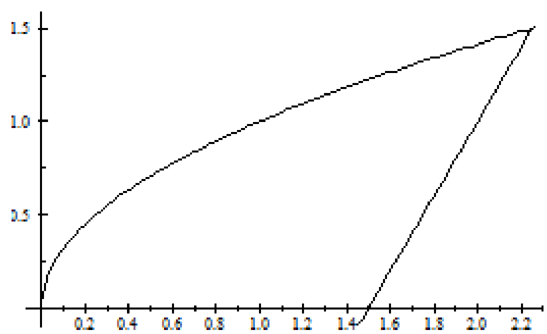
Say R is divided into vertical strips. A typical one determined by an interval $[x_i, x_{i+1}]$ is shaded in the right figure.

(a) *Show that the shaded strip may be approximated by a rectangle with length $4(1 - x_i)$ and width $\triangle x = (x_{i+1} - x_i)$.*

(b) *Estimate the volume of the shell generated by the approximating rectangle when it is rotated about the y-axis through 4 right angles.*

(c) *Calculate the volume of the solid generated by rotating R about the y-axis through 4 right angles.*

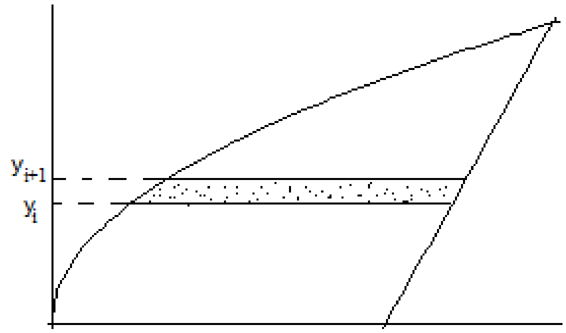
2. *The figure below shows the region in the first quadrant enclosed by the x-axis, the line $y = 2x - 3$ and the curve $y = \sqrt{x}$.*



(a) *Show that the line and the curve intersect at $\left(\frac{9}{4}, \frac{3}{2}\right)$.*

(b) *Say the region is divided into horizontal strips. A typical one determined by an interval $[y_i, y_{i+1}]$*

is shaded in the right figure.

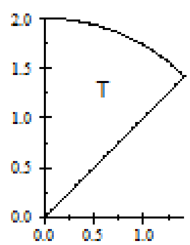


Show that the shaded strip may be approximated by a rectangle with length $\frac{1}{2}(y_i + 3) - y_i^2$ *and width* $\triangle y = (y_{i+1} - y_i)$.

(c) *Estimate the volume of the shell generated by the approximating rectangle when it is rotated about the x-axis through 4 right angles.*

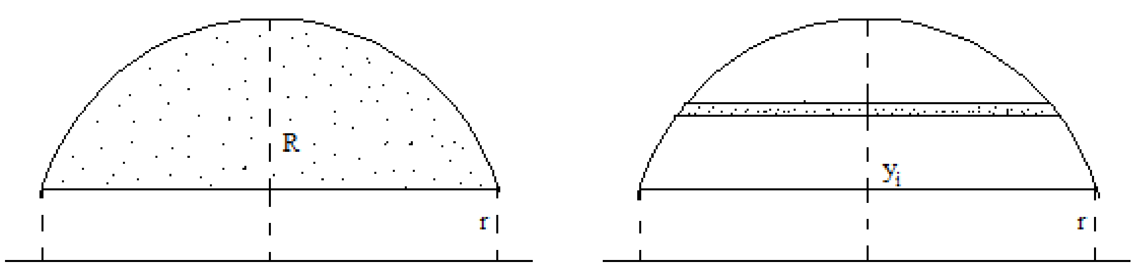
(d) *Calculate the volume of the solid generated by rotating the given region about the x-axis through 4 right angles.*

3. *In the figure below,* T *is the region enclosed by the y-axis, the curve* $y = \sqrt{4 - x^2}$ *and the line* $y = x$.



Use the method of shells to calculate the volume of the solid generated by revolving T *about the y-axis through 4 right angles.*

4. *Consider a sphere of radius* R *and let* $0 < r < R$. *A solid is obtained by drilling, through the center of the sphere, a hole of radius* r. *Another way of obtaining this solid is to revolve, about the x-axis, the shaded region, to the left, in the figure below. The region is obtained by removing, from a half-disc of radius* R, *the section of the half-disc below the line* $y = r$ *You are required to use the method of shells to calculate the volume of solid. The first step is to divide the shaded region into smaller strips parallel to the x-axis then approximate each strip with a suitable rectangle. A typical strip which is* y_i *units above the x-axis is shown in the figure to the right.*



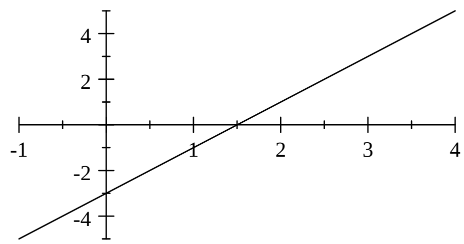
(a) *Show that the typical strip can be approximated by a rectangle of length* $2\left(\sqrt{R^2 - y_i^2}\right)$.

(b) *Calculate the volume of the shell generated by the approximating rectangle then form a Riemann sum.*

(c) *Show that the volume of the solid is* $\frac{4\pi}{3}\left(R^2 - r^2\right)^{3/2}$*, then deduce that the volume of a sphere with radius* R *is* $\frac{4\pi}{3}R^3$*.*

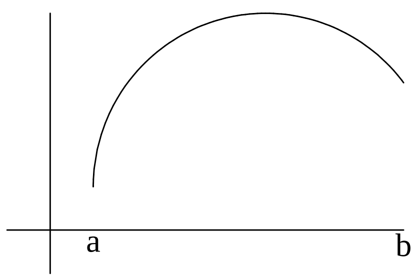
Length of the Graph of a Function on an Interval

If f is a linear function then it is easy to determine the length of a section of its graph using the distance formula. For example, let $f(x) = 2x - 3$. Consider a section of the graph of f between $x = -1$ and $x = 4$. It is the line segment joining $(-1, -5)$ and $(4, 5)$.

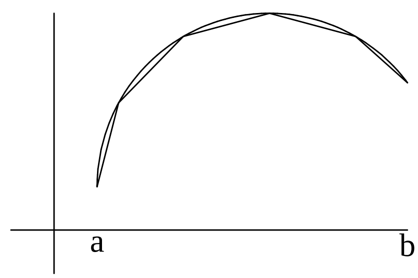


Its length is $\sqrt{(-1-4)^2 + (-5-5)^2} = \sqrt{5^2 + 10^2} = \sqrt{125} = 5\sqrt{5}$ units.

Now consider a non-linear function $f(x)$. Say we wish to determine the length L of the section of its graph between $x = a$ and $x = b$ with $a < b$. Since we have no direct means of calculating lengths of arbitrary curves, we resort to approximating the graph with straight line segments, then use the total length of the line segments to approximate the length of the curve.



Graph between $x = a$ and $x = b$



Graph and five approximating line segments

In the above diagram, the curve is approximated by 5 line segments. In general we would divide the interval $[a, b]$ into n equal subintervals $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_i, x_{i+1}]$, ...,$[x_{n-1}, x_n]$ and approximate the curve with the line segments L_1 joining $(x_0, f(x_0))$ to $(x_1, f(x_1))$, L_2 joining $(x_1, f(x_1))$ to $(x_2, f(x_2))$, ..., L_n joining $(x_{n-1}, f(x_{n-1}))$ to $(x_n, f(x_n))$. The typical line segment joining $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$ has length

$$\sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$$

Therefore the length of the curve is approximately equal to

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} \tag{2}$$

We use the Mean Value theorem to write expression (2) in the form of a Riemann sum. The theorem asserts that if we consider any two values $f(x_{i-1})$ and $f(x_i)$ of f, we can find a number θ_i between x_{i-1} and x_i such that

$$f(x_i) - f(x_{i-1}) = (x_i - x_{i-1})\, f(\theta_i)$$

It follows that the length of the curve is approximately equal to

$$\sum_{i=1}^{n} \sqrt{\left(x_i - x_{i-1}\right)^2 + \left(x_i - x_{i-1}\right)^2 \left(f'(\theta_i)\right)^2} \tag{3}$$

If we factor out $\left(x_i - x_{i-1}\right)^2$ and denote $(x_i - x_{i-1})$ by $\triangle x$, we may write (3) as the Riemann sum

$$\sum_{i=1}^{n} \left(\sqrt{1 + \left(f'(\theta_i)\right)^2}\right)(x_i - x_{i-1}) = \sum_{i=1}^{n} \left(\sqrt{1 + \left(f'(\theta_i)\right)^2}\right) \triangle x$$

The exact length of the curve is the limit of the expression $\sum_{i=1}^{n} \left(\sqrt{1 + \left(f'(\theta_i)\right)^2}\right) \triangle x$ as $\triangle x$ approaches 0. In other words,

$$L = \lim_{\triangle x \to 0} \sum_{i=1}^{n} \left(\sqrt{1 + \left(f'(\theta_i)\right)^2}\right) \triangle x = \int_a^b \sqrt{1 + \left(f'(x)\right)^2} dx$$

Example 9 *Let $f(x) = x^{3/2}$ and L be the length of the graph of f between $x = 1$ and $x = 8$. Then*

$$L = \int_1^8 \sqrt{1 + \left(f'(x)\right)^2} dx = \int_1^8 \sqrt{1 + \tfrac{9x}{4}} dx = \int_1^8 \left(1 + \tfrac{9x}{4}\right)^{1/2} dx$$

We can integrate this by inspection. The result is

$$L = \left[\tfrac{2}{3}\left(1 + \tfrac{9x}{4}\right)^{3/2} \cdot \tfrac{4}{9}\right]_1^8 = \tfrac{8}{27}\left(19^{3/2} - \left(\tfrac{13}{4}\right)^{3/2}\right)$$

Example 10 *Let $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ and L be the length of the graph of f between $x = 1$ and $x = 9$. Then*

$$\begin{aligned} L &= \int_1^9 \sqrt{1 + \left(\tfrac{x}{2} - \tfrac{1}{2x}\right)^2} dx = \int_1^9 \sqrt{\tfrac{x^2}{4} + \tfrac{1}{2} + \tfrac{1}{4x^2}} dx \\ &= \int_1^9 \sqrt{\left(\tfrac{x}{2} + \tfrac{1}{2x}\right)^2} dx = \int_1^9 \left(\tfrac{x}{2} + \tfrac{1}{2x}\right) dx \\ &= \left[\tfrac{x^2}{4} + \tfrac{1}{2}\ln x\right]_1^9 = 20 + \tfrac{1}{2}\ln 9 = 20 + \ln 3 \end{aligned}$$

Exercise 11

1. *Let $f(x) = \frac{1}{3}\left(x^2 + 2\right)^{3/2}$. Show that $1 + (f'(x))^2 = \left(1 + x^2\right)^2$. The length of the graph of f between $x = 0$ and $x = 2$ should be $\int_0^2 \sqrt{\left(1 + x^2\right)^2} dx$. Calculate it.*
2. *Let $f(x) = 2x^{3/2}$. Show that $\sqrt{1 + (f'(x))^2} = \sqrt{1 + 9x}$. Then the length of the graph of f between $x = 0$ and $x = 3$ should be $\int_0^3 \left(\sqrt{1 + 9x}\right) dx$. Calculate it.*
3. *Let $f(x) = \ln(\sec x)$. Show that $1 + (f'(x))^2 = \sec^2 x$, then calculate the length of the graph of f between $x = 0$ and $x = \frac{\pi}{3}$.*
4. *Let $f(x) = \cosh x$. Show that $1 + (f'(x))^2 = 1 + \sinh^2 x = \cosh^2 x$, then calculate the length of the graph of f between $x = -1$ and $x = 3$.*
5. *Calculate the length of the graph of $f(x) = \frac{1}{2}x^2$ between $x = 0$ and $x = \sqrt{3}$.*

6. *A curve in the plane may be described by stating its x and y components as functions of some variable t. (For example, the circle with center (a, b) and radius r may be described as the set of all points (x, y) such that $x = a + r\cos t$ and $y = b + r\sin t$, $0 \le t < 2\pi$.) Consider such a curve C consisting of all the points (x, y) such that $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f and g have derivatives on $[a, b]$. Let L be its length. Partition $[a, b]$ into n equal subintervals $[t_0, t_1]$, $[t_1, t_2]$, $\ldots$, $[t_{n-1}, t_n]$. The line segments joining $(f(t_{i-1}), g(t_{i-1}))$ to $(f(t_i), g(t_i))$ for $i = 1, \ldots, n$ may be used to approximate the length L of the curve.*

   (a) *Show that $L \simeq \sum_{i=1}^{n} \left(\sqrt{(f(t_i) - f(t_{i-1}))^2 + (g(t_i) - g(t_{i-1}))^2} \right)$ then deduce that*

   $$L = \int_a^b \sqrt{(f'(t))^2 + g'(t))^2}dt.$$

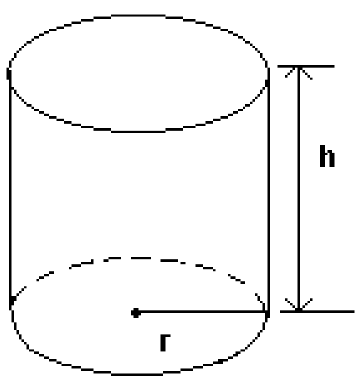
   (b) *Use the above result to calculate the circumference of the circle $x = a + r\cos t$, $y = b + r\sin t$, $0 \le t \le 2\pi$.*

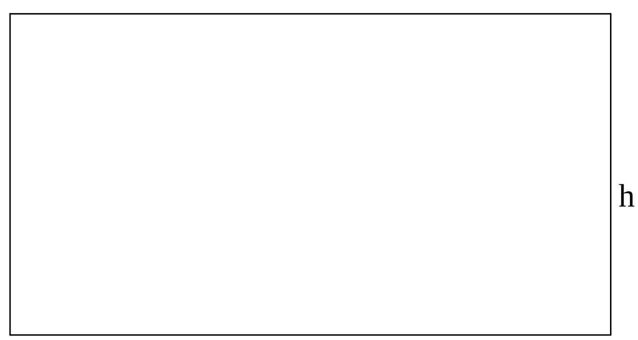
   (c) *Use the result of part (a) to write down an expression for the length of an ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$. What you get is an example of an elliptic integral.*

Area of a Surface of Revolution

Take the graph of a function f on an interval $[a, b]$ and revolve it about a given axis through 4 right angles. What you get is called a surface of revolution. This section addresses the problem of determining the areas of such surfaces.

Example 12 *When you revolve a line segment of length h about an axis parallel to the line segment, you get a cylinder. The radius of the cylinder is the distance r between the line segment and the axis of rotation.*

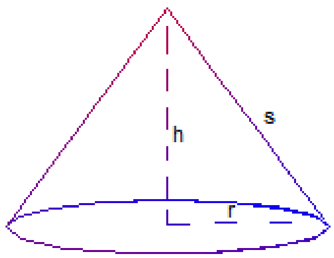




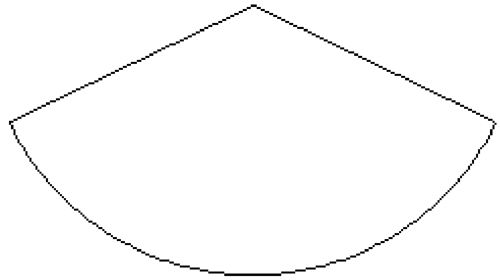
circumference of cylinder

To calculate the surface area of the above cylinder, imagine cutting the figure along a line in the surface parallel to the axis of rotation. When you open it out, you get a rectangle. One side of the rectangle is the circumference of the cylinder. The other side is its height. The area of the rectangle, which is $2\pi rh$ square units, is the surface area of the cylinder.

Example 13 *When you revolve a line segment of length s about an axis making a non-zero angle with the line segment, you get a right circular cone. The cone in the diagram below was obtained by revolving, about the y-axis, a line that makes a non-zero angle with the y-axis. The radius r of its base is called the* ***radius of the cone***. *The distance h from the center of its base to its tip is called the* ***height of the cone***. *The* ***slant height of the cone*** *is the shortest distance s from the tip of the cone to the circumference of the base. To calculate the surface area of the curved part of the cone, note that if you cut the cone along a slant line* *and open out, you get a sector of a circle, (shown below), with radius s and arc length $2\pi r$. Let the angle of the sector be θ. Since $s\theta = 2\pi r$, it follows that $\theta = \frac{2\pi r}{s}$.*



A cone

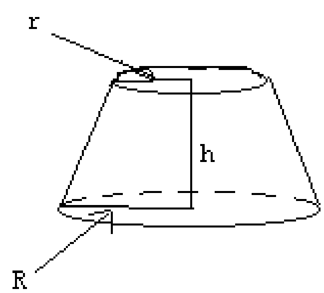


Resulting sector of a circle

The area of the sector, which is $\frac{1}{2}s^2\theta$, is the surface area of the curved part of the cone. It follows that the surface area of the cone is

$$\frac{1}{2}s^2\theta = \frac{1}{2}s^2 \cdot \frac{2\pi r}{s} = \pi rs \text{ square units.} \tag{4}$$

To calculate the surface areas of a number of surfaces, we approximate them with **frustums of a cone**. A frustum of a cone is a figure, like the one below, obtained by chopping off a right circular cone from a given right circular cone.



A frustum of a cone

We need an expression for its surface area. Let the radius of its bigger face be R and that of the smaller face be r. Let the slant height of the cone that is chopped off be l_1 and the slant height of the original cone (before the smaller cone is cut off) be l_2. Then the slant height l of the frustum is given by the equation $l_2 = l + l_1$, and the area of the curved part of the frustum is

$$\pi Rl_2 - \pi rl_1 = \pi R(l_1 + l) - \pi rl_1 = \pi (R - r) l_1 + \pi Rl$$

To get rid of the term $\pi (R - r) l_1$, we use the fact that $\frac{l_1}{r} = \frac{l_1 + l}{R}$. The result is $(R - r) l_1 = rl$, which implies that $\pi (R - r) l_1 = \pi rl$. Therefore the area of the curved part of the frustum is

$$\pi rl + \pi Rl = \pi l (R + r) \text{ square units.}$$

Example 14

1. *The area of the curved surface of a cone with base radius 3 cm and slant height 7 cm is 21π square centimeters.*

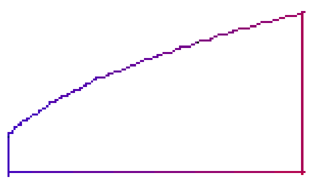
2. *The area of the curved surface of a cone with base radius 4 cm and height 6 cm is*

$$\pi(4)(\sqrt{4^2+6^2}) = 8\sqrt{13}\pi \text{ square centimeters.}$$

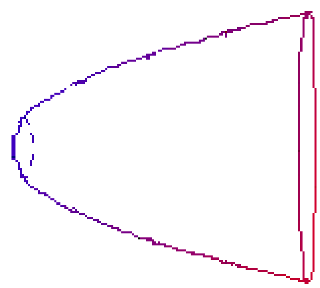
3. *The area of the curved surface of a frustum with slant height 10 cm, faces of radii 14 cm and 24 cm respectively is*

$$\pi(10)(14+24) = 380\pi \text{ cubic centimeters}$$

We can now derive an expression for the area of a surface of revolution. To this end, consider the graph of some function f on an interval $[a, b]$ and the surface generated by revolving the graph about the x-axis through 4 right right angles.

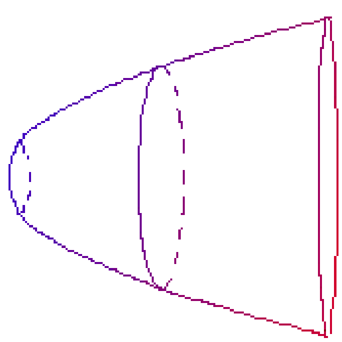


Graph of f

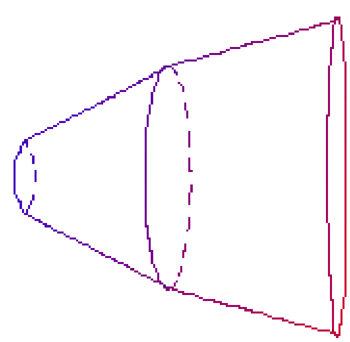


Surface of revolution

To calculate an approximate value of the area of revolution, we partition it into n segments by dividing $[a, b]$ into n equal subintervals $[x_0, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$ of length $\triangle x = h = \frac{1}{n}(b-a)$ each. Each subinterval gives a segment of the area. We approximate each segment by a frustum . The left figure below shows 2 segments that partition the surface. The figure to the right gives the frustums approximating the 2 segments.



Surface partitioned into 2 segments



Approximating frustums

A typical segment on an interval $[x_{i-1}, x_i]$ is approximated by a frustum with faces of radii $f(x_{i-1})$ and $f(x_i)$. Its slant height is $s = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}$. Therefore its area is

$$\pi\left(\sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}\right)[f(x_i) + f(x_{i-1})] \tag{5}$$

There is a point θ_i between x_{i-1} and x_i such that $f(x_i) - f(x_{i-1}) = f'(\theta_i)(x_i - x_{i-1})$, (by the Mean Value Theorem for derivatives), hence we may write (5) as

$$\pi\left(\sqrt{1 + [f'(\theta_i)]^2}\right)[f(x_i) + f(x_{i-1})](x_i - x_{i-1})$$

Therefore an approximation of the required area is

$$\pi \sum_{i=1}^{n} \left(\sqrt{1 + [f'(\theta_i]^2} \right) [f(x_i) + f(x_{i-1})] (x_i - x_{i-1}) \qquad (6)$$

Because x_i, x_{i-1} and θ_i are not the same point, (6) is not a Riemann sum. To get around this, assume that f is a continuous function. Then when $x_i - x_{i-1}$ is small, the numbers $f(x_i)$ and $f(x_{i-1})$ may be approximated by $f(\theta_i)$. Therefore the area of the surface of revolution is approximately equal to

$$2\pi \sum_{i=1}^{n} \left(\sqrt{1 + [f'(\theta_i]^2} \right) f(\theta_i) (x_i - x_{i-1}) = 2\pi \sum_{i=1}^{n} \left(\sqrt{1 + [f'(\theta_i]^2} \right) f(\theta_i) \triangle x$$

The exact area is the limit of these Riemann sums as $\triangle x \to 0$. In other words,

$$Area = 2\pi \int_a^b \left(\sqrt{1 + [f'(x)]^2} \right) f(x) dx \qquad (7)$$

Example 15 *Let $f(x) = \cosh x$ and consider the surface generated when the graph of f between $x = 0$ and $x = 3$ is revolved about the x-axis through 4 right angles. Let its area be A. Since $1 + \sinh^2 x = \cosh^2 x$, and $\cosh^2 x = \cosh 2x + 1$,*

$$\begin{aligned} A &= 2\pi \int_0^3 \left(\sqrt{1 + \sinh^2 x} \right) \cosh x dx = 2\pi \int_0^3 \cosh^2 x dx \\ &= 2\pi \int_0^3 \tfrac{1}{2} (\cosh 2x + 1) \, dx = \pi \left[\frac{\sinh 2x}{2} + x \right]_0^3 = \pi \left(\frac{\sinh 6}{2} + 3 \right) \end{aligned}$$

Exercise 16

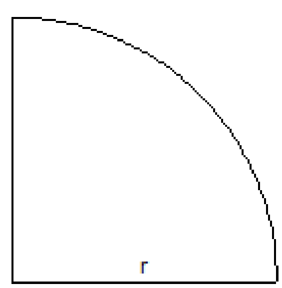
1. *Let $f(x) = \frac{1}{2}x$. Consider the graph of f between $x = 0$ and $x = 2$. When we revolve this graph about the x-axis through 4 right angles, we get a cone. Calculate the area of its curved surface using formula (7) and verify that the answer you get is in agreement with the value given by formula (4) on page 18.*

2. *Let $f(x) = a\sqrt{x}$ where a is a positive constant. Show that:*

   (a) $\left(\sqrt{1 + [f'(x)]^2} \right) f(x) = \dfrac{a\sqrt{a^2 + 4x}}{2}$.

   (b) *The area of the surface generated by revolving the graph of f between $x = 0$ and $x = a^2$, about the x-axis through 4 right angle is $\frac{\pi a^4}{6}(5^{3/2} - 1)$.*

3. *A hemisphere with radius r may be obtained by rotating the graph of $f(x) = \sqrt{r^2 - x^2}$, $0 \leq x \leq r$, about the x-axis through 4 right angles.*



Use the formula $Area = 2\pi \int_a^b \left(\sqrt{1 + [f'(x)]^2} \right) f(x) dx$ to confirm that the surface area of a sphere with radius r is $4\pi r^2$ square units.

4. *Consider the graph of $f(x) = \sin x$ on the interval $[0, \pi]$. Let A be the area of the surface obtained by revolving the graph about the x-axis through 4 right angles. Show that $A = \int_0^{\pi} \left(\sqrt{1+\cos^2 x}\right) \sin x dx$, then use the substitution $u = \cos x$ to deduce that $A = \int_{-1}^{1} \left(\sqrt{1+u^2}\right) du$. We have, so far, introduced two methods of evaluating this integral. Evaluate it using any one of them.*

Work Done by a Force

Work is done when a force is exerted on an object and causes it to move from one point to another. For example, work is done when:

1. A student lifts a backpack full of books from the floor and places it on his shoulder.
2. A weight-lifter lifts a barbell above his head.
3. A horse pulls a plow through a field.

By definition, if a constant force of *magnitude* F is applied to an object and causes it to move a *distance* d *in the direction of the force* then the work W done by the force is given by the formula

$$W = F \times d$$

The following examples illustrate how we use Riemann sums to calculate work done by non-constant forces:

Example 17 *Consider the swimming pool shown in the Figure 1 below. It has a shallow side that is 2 meter deep and a deep (opposite) side that is 8 meters deep. It is 7 meters wide and 12 meters long.*

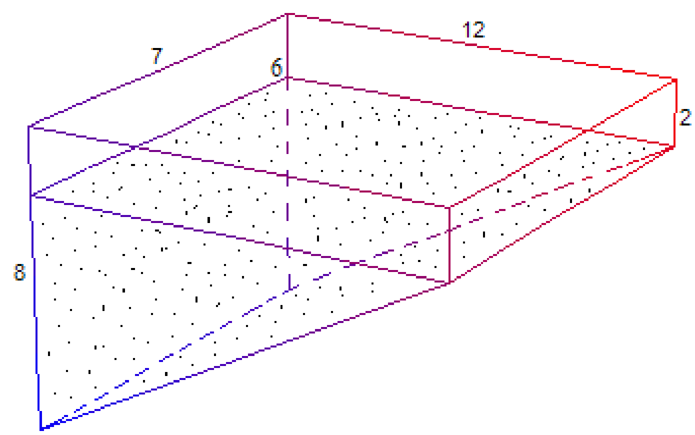


Figure 1.

It contains water that comes up to the 6 meter mark of the deep end of the pool. We wish to calculate the work done to pump all the water out. To this end, imagine partitioning the water into thin layers, by dividing the interval $[0, 6]$ into n equal subintervals of length $6/n$ meters each. Each interval determines a thin layer *of the water. A typical one, determined by an interval $[z_i, z_{i+1}]$, is drawn in the Figure 2 below.*

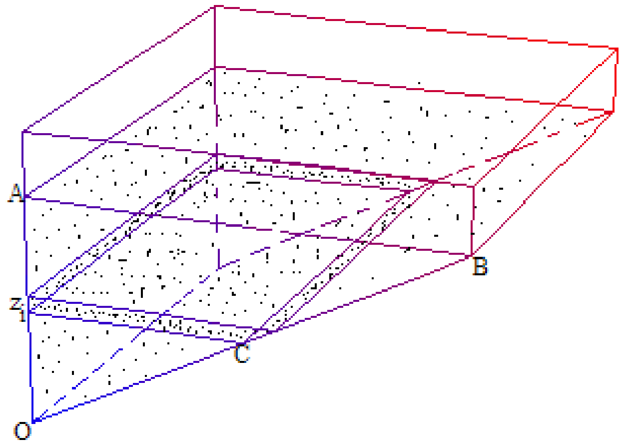


Figure 2.

It is also shown magnified in Figure 3 below.

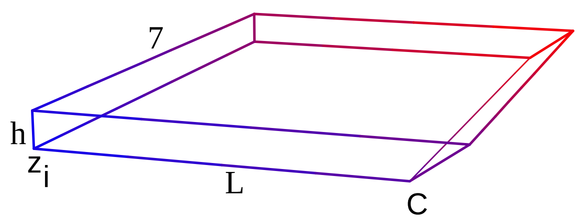


Figure 3.

We approximate it with a rectangular box of length L, width 7 and thickness $h = \frac{6}{n}$*. Using the similar triangles* Oz_iC *and OAB in Figure 2, the length L of* Cz_i *is given by*

$$\frac{L}{12} = \frac{z_i}{6}.$$

It follows that $L = 2z_i$ *meters, therefore its volume is approximately*

$$(7)(2z_i)(h) = 14z_ih \text{ cubic meters.}$$

Since a cubic meter of water weighs 1000 kilograms, the thin layer weighs approximately $14000z_ih$ *kilograms. It has to be lifted through a distance of* $(8 - z_i)$ *meters to the surface of the pool. The work done is approximately*

$$14000z_ih(8 - z_i) \text{ kilogram wt. meters}$$

An estimate of the total work done to empty the tank is obtained by adding the work done to move all the n layers. If the work is W then

$$W \simeq 14000z_0h(8 - z_0) + \cdots + 14000z_{n-1}h(8 - z_{n-1}) = \sum_{i=0}^{n-1} 14000z_ih(8 - z_i)$$

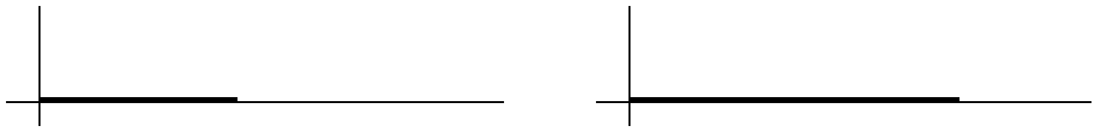
This is a Riemann sum of the function

$$f(z) = 14000z(8 - z) = 112000z - 14000z^2.$$

The exact value of W is the limit of $\sum_{i=0}^{n-1}\left(112000z_i - 14000z_i^2\right)h$ *as* $h \to 0$*. That limit is*

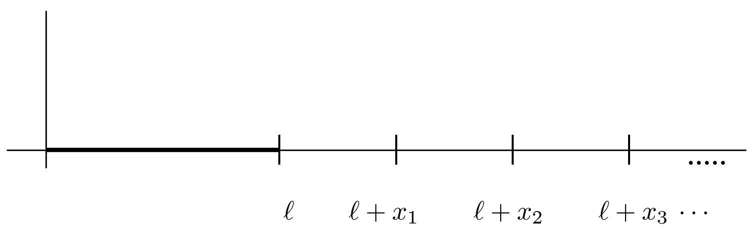
$$\int_0^6 (112000z - 14000z^2)dz = \left[56000z^2 - \frac{14000}{3}z^3\right]_0^6 = 1008000 \text{ Kg. m}$$

Example 18 *Imagine an elastic string on the x-axis with one end fixed to an object at the the origin $(0,0)$ and the other end free to move along the horizontal axis. Its length when it is not stretched is called the natural length of the string and we denote it by ℓ. Assume that when it is not stretched, the free end is at $(\ell, 0)$.*



Un-stretched string *Stretched string*

It is known from experiments that when the free end is pulled a distance x to a point $(\ell + x, 0)$ the string exerts a force, (a pull), of magnitude kx where k is a constant, (called the constant of the string). Assume that force is measured in kilogram weight and distances in meters. Imagine pulling the free end from the point $(\ell, 0)$ all the way to a point $(\ell + b, 0)$ where $b > 0$. We wish to calculate the work done in doing so. The force you have to exert to pull it starts off small, when x is small, and increases as x increases. Therefore the work done it is ***not*** *$(bk) \cdot (b) = kb^2$, which one obtains when one multiplies the force kb (when the string is stretched b units) by the distance b through which the free end of the string is pulled. It would be kb^2 if the force were constant and equal to bk all the way from the start to finish, but that is not the case. To deal with the changing force, we divide the interval $[0, b]$, (because the stretching changes from 0 to b), into n equal subintervals of length $h = \frac{b}{n} = \triangle x$ each. They are $[x_0, x_1] = \left[0, \frac{b}{n}\right]$, $[x_1, x_2] = \left[\frac{b}{n}, \frac{2b}{n}\right]$, $[x_2, x_3]$, $\ldots$, $[x_{n-1}, b]$.*



Now imagine pulling the free end in stages: from $(\ell, 0)$ to $(\ell + x_1, 0)$, then from $(\ell + x_1, 0)$ to $(\ell + x_2, 0)$, and so on. In the last stage, we pull it from $(\ell + x_{n-1}, 0)$ to $(\ell + x_n, 0) = (\ell + b, 0)$. To estimate the work done in pulling it from $(\ell + x_i, 0)$ to $(\ell + x_{i+1}, 0)$, we assume that the force needed to pull it is constant and equal to $f(x_i) = kx_i$. In particular, we assume that the force needed to pull it from $(\ell, 0)$ to $(\ell + x_1, 0)$ is 0; the force needed to pull it from $(\ell + x_1, 0)$ to $(\ell + x_2, 0)$ is kx_1, and so on. Of course these are all approximations, because the force does not remain constant. But they are good approximations when h is very small. It follows that the work done to stretch it by a length of b units from its natural length is approximately equal to

$$hf(x_0) + hf(x_1) + \cdots + hf(x_{i-1}) = \sum_{i=0}^{n-1} hf(x_i) \tag{8}$$

Let its exact value be W. Then W should be the limit of the sums (8) as $h \to 0$. In other words

$$W = \lim_{h \to 0} \sum_{i=0}^{n-1} hf(x_i) = \lim_{\triangle x \to 0} \sum_{i=0}^{n-1} f(x_i) \triangle x = \int_0^b f(x) dx$$

Since $f(x) = kx$ which has antiderivative $F(x) = \frac{1}{2}kx^2$, it follows that

$$W = \left[\frac{1}{2}kx^2\right]_0^b = \frac{1}{2}kb^2.$$

Exercise 19

1. *A chain weighs 1 kilogram per meter, and it hangs from a point on the roof of a house that is 20 meters from the ground.*

(a) *What is the work done to pull such a 20 - meter chain to the roof?*

(b) *What is the work done to pull such a 28 - meter chain to the roof?*

2. *The fuel tank for a tractor trailer is a cylinder lying on its curved side. It has radius r and length L meters. It contains fuel of density d kg per cubic meter. The fuel pump pumps the fuel to a point in the engine where it is ignited. The point is H meters above the highest point on the tank. Suppose the tank is half full.*

   (a) *Show that the work done, in kilogram meters, by the pump to pump all the fuel into the engine is*

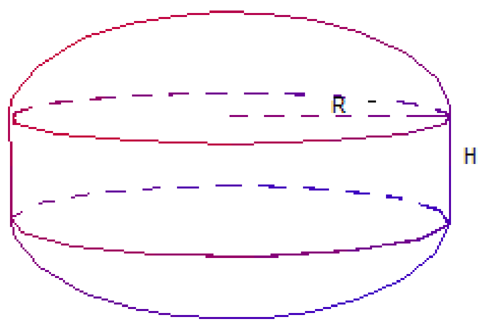
$$2dL\int_0^r \left[r^2-(r-z)^2\right]^{1/2}\left[H+(r-z)\right]dz$$

   (b) *Show that the substitution $u=(r-z)$ reduces the above integral to*

$$2dL\int_0^r \left(r^2-u^2\right)^{1/2}(H+u)\,du$$

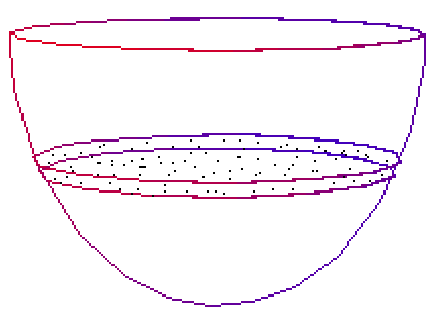
   then evaluate it.

3. *A water storage tank was made by welding a hemisphere onto each of the two ends of a cylinder of radius $R=8$ meters and height $H=6$ meters respectively. (There are many such tanks along highways.)*



It is supported upright by a metal structure in such a way that its lowest point is 50 meters above the ground. Your are required to calculate the work done to fill it up by pumping water from the ground level. Assume that 1 cubic meter of water weighs 1000 Kg.

   (a) *Start by calculating the work done to fill the lower hemisphere of the tank. To do so, partition the water in the lower hemisphere into smaller portions by dividing the interval $[0,8]$ into n smaller subintervals $[z_0,z_1]$, $[z_1,z_2]$, ..., $[z_{n-1},z_n]$ where $z_0=0$ and $z_n=8$. These intervals determine smaller portions of the water. A typical one determined by an interval $[z_i,z_{i+1}]$ is shaded in the figure below showing an exaggerated lower hemisphere.*



Approximate this portion of the water with a disc of radius $\sqrt{8^2-(8-z_i)^2}$ *, thickness* $\triangle z = z_{i+1} - z_i$ *and weight* $\pi\left(\sqrt{8^2-(8-z_i)^2}\right)^2 \triangle z$ *kilograms. Imagine lifting it from the ground level to where it is now. Calculate the work done and deduce that the work done, in kilogram wt. meters, to fill the lower hemisphere is*

$$\pi\int_0^8 \left(16z - z^2\right)(50+z)\,dz$$

then evaluate the definite integral.

(b) *Calculate the work done to fill up the cylindrical part, and the work done to fill up the top hemisphere then add up to get the total work done to fill up the tank.*

Force Exerted by a Fluid

We use an example to show how forces may be calculated using Riemann sums:

Example 20 *Imagine a rectangular tank of length* L *and width* W *containing a fluid of uniform density* u *to height* H *units as shown in figure (i).*

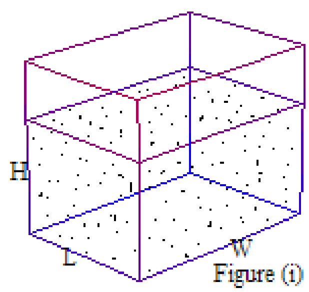


Figure (i)

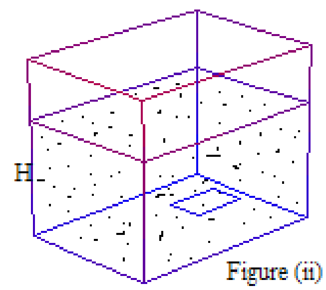


Figure (ii)

Imagine a rectangular hole, shown in figure (ii), in the base of the tank. The hole is plugged with a rectangular object of the same size. A force has to be exerted on the plug to stop it from popping out. It turns out, (these are experimental facts from physics), that the fluid force per unit area at any point on the plug is equal to

(Density of fluid) $\times$ *(height* H *of column of fluid above point).*

Therefore the total force on the plug, (we let you figure out the units), is

(Density of fluid) $\times$ *(height* H *of column of fluid above point)* $\times$ *(area of plug).*

This implies that the total force on the base of the tank is

(Density of fluid) $\times$ *(height* H *of column of fluid above point)* $\times$ *(area of base).*

Now imagine a hole in a side of the tank as shown in figure (iii) below.

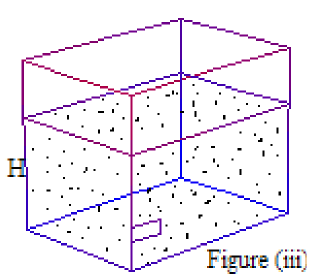


Figure (iii)

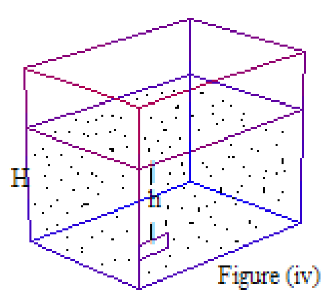


Figure (iv)

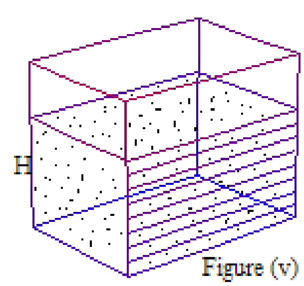


Figure (v)

It is plugged like the hole in the bottom. It is also an experimental fact that the force per unit area at a point on this plug is equal to

$$\text{(Density of fluid)} \times \text{(vertical distance } h \text{ from the point to the surface of fluid).}$$

The distance h is shown in figure (iv). This time the force per unit area is not the same at all the points of the plug because h may change from one point to another. To estimate the force on the plug, assume that it is very thin. Then it is reasonable to approximate the force per unit area at an arbitrary point on the plug with the force per unit area at the top end of the plug. Therefore the force needed to keep it in place is approximately equal to

$$\text{(Density of fluid)} \times \text{(distance } h \text{ from top of plug to surface of fluid)} \times \text{(area of plug).}$$

Say we wish to calculate the force on the face in figure (iii) with the plug. We divide the face into n small strips of thickness $\triangle h$ each as shown in figure (v), estimate the force on each strip, add up to get an estimate of the total force on the face then take limits. The force on a strip whose center is h_i units from the surface of the fluid is $uh_iW\triangle h$. The total force on the face is approximately

$$\sum_{i=1}^{n} uh_iW\triangle h.$$

The exact force is the limit as $\triangle h \to 0$ of the above Riemann sum. That limit is

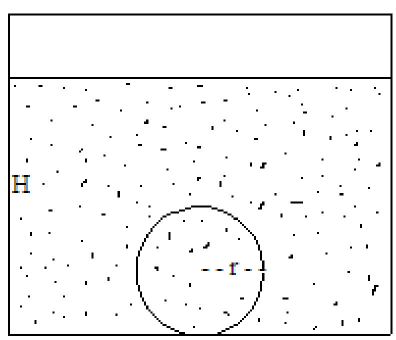
$$\int_0^H uWh\,dh = \frac{uWH^2}{2}.$$

Exercise 21

1. *A rectangular tank contains a liquid with density u kilogram per cubic meter to a depth of H meters. A circular plate with radius r meters, ($r < \frac{1}{2}H$), is in one of the sides of the tank as shown in the figure below. Show that the fluid force acting on the plate is*

$$\int_0^{2r} \left[r^2 - (r-z)^2\right] [H - z]\, dz$$

then evaluate the integral.

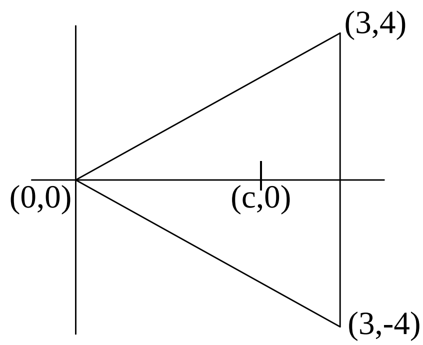


2. *A cylindrical tank with closed top and base has radius r and height H. It is half full with a fluid of density u. Imagine laying it on its curved side. What is the force on each of the two circular ends?*

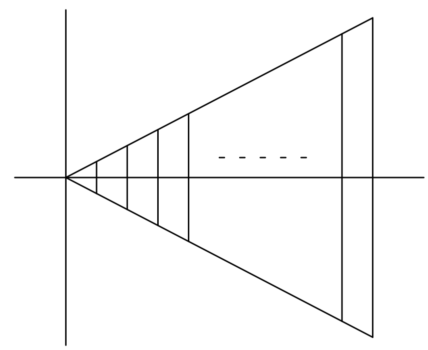
3. *A dam is 300 m long and 40 m high. The density of water is 1000 kg per cubic meter. Calculate the force of water on the dam when the water level behind it is 25 m.*

Center of Gravity

Example 22 *Consider a uniform triangular lamina with vertices at* $(0,0)$, $(3,4)$ *and* $(3,-4)$. *It is the region enclosed by the lines* $f(x) = \frac{4}{3}x$, $g(x) = -\frac{4}{3}x$ *and the line* $x = 3$. *We wish to determine its center of gravity. This is the point where the lamina balances on a pin-head.*



Given lamina



Lamina divided into small segments

Since the lamina is symmetric about the x-axis, the center of gravity must be on the x-axis. Let it be at the point $(c,0)$. *Then the moment of the lamina about the origin is* Mc *where* M *is its weight. Indeed* $M = w \cdot 4 \cdot 3 = 12w$ *where* w *is the weight per unit area of the lamina, therefore the moment of the lamina about the origin is*

$$Mc = 12wc$$

We need another expression for the moment of the lamina about the origin in order to solve for c. *To get it, we use the fact that the moment of a body is equal to the sum of the moments of its parts. We therefore divide the lamina into smaller segments, as shown above, estimate the moment of each segment and total up. Say we divide it into* n *segments of thickness* $\triangle x = \frac{3}{n}$ *each. A typical segment is between* $x_i = i \cdot \frac{3}{n}$ *and* $x_{i+1} = (i+1) \cdot \frac{3}{n}$. *Its area is approximately equal to* $2f(x_i)\triangle x$, *therefore its moment about the origin is approximately*

$$2wf(x_i)\triangle x \cdot x_i = 2wf(x_i)x_i\triangle x$$

It follows that the moment of the lamina about the origin is approximately equal to

$$\sum_{i=0}^{n-1} 2wf(x_i)x_i\triangle x$$

The exact moment is $\displaystyle\lim_{\triangle x \to 0} \sum_{i=0}^{n-1} 2wf(x_i)x_i\triangle x = 2w\int_0^3 xf(x)dx = \frac{8w}{3}\int_0^3 x^2dx$. *We have used the fact that* $f(x) = \dfrac{4x}{3}$. *We easily evaluate the definite integral to get*

$$\frac{8w}{3}\int_0^3 x^2dx = \left[\frac{8w}{9}x^3\right]_0^3 = 24w$$

Equating the two expressions $12wc$ *and* $24w$ *for the total moment about the origin gives* $c = 2$. *Thus the lamina balances on a pin-head at* $(2,0)$.

Exercise 23

1. *Find the position of the center of gravity of a uniform lamina in the shape of a half disc of radius* r.
2. *A uniform lamina has the shape of a trapezium with vertices at* $(-3,0)$, $(3,0)$, $(2,4)$ *and* $(-2,4)$. *Calculate its center of gravity. (Imagine dividing the lamina into small horizontal segments of width* $\triangle y$. *Estimate the moment of a typical segment about the x-axis then form a Riemann sum.)*

3. *Find the position of the center of gravity of a uniform lamina in the shape of the region enclosed by the graph of* $f(x) = x^2$ *and the line* $y = 4$.

4. *An isosceles triangle has vertices at* $A(-a, 0)$, $B(a, 0)$ *and* $C(0, b)$ *where* $b > 0$. *Let* $O(0, 0)$ *be the origin. Show that its center of gravity is one third of the way up the line* OC.