Integration by Partial Fractions

Say you have to determine $\int\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx$. You would immediately integrate each term:

$$\int\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx=-\ln|1-x|+\ln|1+x|+c=\ln\left|\frac{1+x}{1-x}\right|+c$$

However,

$$\frac{1}{1-x}+\frac{1}{1+x}=\frac{2}{1-x^2},$$

therefore you could have been asked, instead, to determine $\int\frac{2}{1-x^2}dx$. In this form, you would probably try a substitution like $x=\sin u$, to get

$$\int\frac{2}{1-x^2}dx=2\int\frac{\cos u}{\cos^2 u}du=2\int\sec u du=2\ln|\sec u+\tan u|+c$$

You would have to do some extra work to express $2\ln|\sec u+\tan u|+c$ in terms of x. Example:

$$\begin{aligned}2\int\sec u du &= 2\ln|\sec u+\tan u|+c=2\ln\left|\frac{1}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}\right|+c\\ &= 2\ln\left|\frac{1+x}{\sqrt{1-x^2}}\right|+c=2\ln\left|\frac{\sqrt{1+x}}{\sqrt{1-x}}\right|+c=\ln\left|\frac{1+x}{1-x}\right|+c\end{aligned}$$

A more complicated integral like

$$\int\frac{4x+2}{(x^2-1)(x+2)}dx$$

may not yield to the familiar substitutions we have encountered. But

$$\frac{4x+2}{(x^2-1)(x+2)}=\frac{1}{x+1}-\frac{2}{x-1}+\frac{3}{x+2}$$

and if the integral is given, instead, as

$$\int\left(\frac{1}{x+1}-\frac{2}{x-1}+\frac{3}{x+2}\right)dx$$

you would easily handle it. These examples suggest that given an integral of a rational function, it may be a good idea to split the integrand into partial fractions then integrate. Techniques of decomposing rational functions into partial fractions are developed in most pre-calculus textbooks. Here we consider only a few typical examples.

Example 1 *To determine* $\int\frac{x}{(1-x)(x-3)(x+2)}dx$*, we split the integrand as*

$$\frac{x}{(1-x)(x-3)(x+2)}=\frac{A}{1-x}+\frac{B}{x-3}+\frac{C}{x+2}$$

where A, B and C are constants. Solving gives $A=-\frac{1}{6}$, $B=-\frac{3}{10}$ and $C=\frac{2}{15}$. Therefore

$$\begin{aligned}\int\frac{x}{(1-x)(x-3)(x+2)}dx &= -\frac{1}{6}\int\frac{1}{1-x}dx-\frac{3}{10}\int\frac{1}{x-3}dx+\frac{2}{15}\int\frac{1}{x+2}dx\\ &= \tfrac{1}{6}\ln|1-x|-\tfrac{3}{10}\ln|x-3|+\tfrac{2}{15}\ln|x+2|+c\end{aligned}$$

Example 2 *To determine* $\int \frac{1}{(1-x)(x+2)^2}dx$, *we split the integrand as*

$$\frac{1}{(1-x)(x+2)^2} = \frac{A}{1-x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

where A, B *and* C *are constants. Solving gives* $A = \frac{1}{9}$, $B = \frac{1}{9}$ *and* $C = \frac{1}{3}$. *Therefore*

$$\begin{aligned}\int \frac{1}{(1-x)(x+2)^2}dx &= \frac{1}{9}\int \frac{1}{1-x}dx + \frac{1}{9}\int \frac{1}{x+2}dx + \frac{1}{3}\int \frac{1}{(x+2)^2}dx \\ &= \tfrac{1}{9}\ln\left|\frac{x+2}{1-x}\right| - \frac{1}{3(x+2)} + c\end{aligned}$$

Example 3 *To determine* $\int \frac{6}{(2x^2+1)(x-1)}dx$, *we split the integrand as*

$$\frac{6}{(2x^2+1)(x-1)} = \frac{Ax+B}{2x^2+1} + \frac{C}{x-1}$$

where A, B *and* C *are constants. Solving for* A, B *and* C *yields* $A = B = -4$ *and* $C = 2$. *Write* $\frac{6dx}{(2x^2+1)(x-1)}$ *as* $-\frac{4x}{2x^2+1} - \frac{4}{2x^2+1} + \frac{2}{x-1}$. *Then*

$$\begin{aligned}\int \frac{6dx}{(2x^2+1)(x-1)} &= -\int \frac{4xdx}{2x^2+1} - 4\int \frac{dx}{2x^2+1} + 2\int \frac{dx}{x-1} \\ &= -\ln\left(2x^2+1\right) - \tfrac{4}{\sqrt{2}}\arctan\sqrt{2}x + 2\ln|x-1| + c\end{aligned}$$

Exercise 4

1. *Integrate by partial fractions. In part (b), k is a non-zero constant.*

   (a) $\int \frac{xdx}{(x-1)(x+2)}$ *b)* $\int \frac{dx}{x^2-k^2}$ *(c)* $\int \frac{dx}{x^2(x+1)}$

   (d) $\int \frac{\left(4x^2+6x-3\right)dx}{x^3+2x^2-3x}$ *e)* $\int \frac{(x-2)dx}{x^2(x-1)^2}$ *(f)* $\int \frac{x^2dx}{x^4-16}$

   (g) $\int \frac{dx}{x+\sqrt{x}-2}$, *(Let* $u = \sqrt{x}$.*)*   *h)* $\int \frac{dx}{x^2(x^2+2)}$ *(i)* $\int \frac{x}{2x+3\sqrt{x}+1}$

2. *If the degree of the numerator of a rational integrand is not less than the degree of the denominator, first do a long division to get a numerator of lower degree than that of the denominator. For example,given* $\int \frac{x^3+x^2-3}{x^2-1}dx$, *we first do a long division to get*

$$\frac{x^3+x^2-3}{x^2-1} = x+1+\frac{x-2}{x^2-1}$$

   We then split $\frac{x-2}{x^2-1}$ *into partial fractions. The result is*

$$\frac{x-2}{x^2-1} = \frac{x-2}{(x-1)(x+1)} = \frac{3}{2(x+1)} - \frac{1}{2(x-1)}$$

Therefore

$$\int \frac{x^3+x^2-3}{x^2-1}dx = \int \left(x+1+\frac{3}{2(x+1)}-\frac{1}{2(x-1)}\right)dx$$
$$= \tfrac{1}{2}x^2+x+\tfrac{3}{2}\ln|x+1|-\tfrac{1}{2}\ln|x-1|+c$$

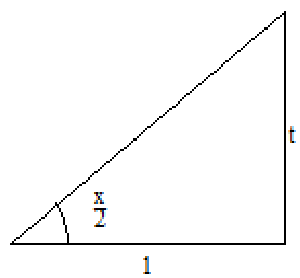
Determine the following in a similar way

$$a) \int \frac{x^3+x+1}{(x+1)(x-3)}dx \quad b) \int \frac{x^4}{x^3-1}dx \quad c) \int \frac{x^4+1}{x^4-1}dx \quad d) \int \frac{x^3-1}{x^3+1}dx$$

The substitution $t=\tan\frac{x}{2}$

The substitution $t=\tan\frac{x}{2}$ enables us to write $\sin x$ and $\cos x$ as rational functions of the new variable t. We are then able to transform an integrand involving rational functions of $\cos x$ and $\sin x$ into an integrand involving rational functions of t. The latter integrand may be split into partial fractions which we know how to handle. The following are the details for re[placing $\sin x$ and $\cos x$ by rational expressions involving t.

Let $t=\tan\frac{x}{2}$. The figure below shows a right triangle with such an angle $\frac{x}{2}$.



Its hypotenuse has length $\sqrt{1+t^2}$, therefore $\sin\frac{x}{2}=\dfrac{t}{\sqrt{1+t^2}}$ and $\cos\frac{x}{2}=\dfrac{1}{\sqrt{1+t^2}}$. We now appeal to the trigonometric identities $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}$ to write $\sin x$ and $\cos x$ in terms of t. The result is

$$\sin x = 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)=\frac{2t}{1+t^2},$$

and

$$\cos x = \left(\frac{1}{\sqrt{1+t^2}}\right)^2-\left(\frac{t}{\sqrt{1+t^2}}\right)^2=\frac{1-t^2}{1+t^2}$$

The change of variables also demands that we replace dx by an expression involving dt. Since $\dfrac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}$, we proceed to solve for dx and the result is

$$dx=\frac{dt}{\frac{1}{2}\sec^2\frac{x}{2}}=2\left(\cos^2\tfrac{x}{2}\right)dt=\left(\frac{2}{1+t^2}\right)dt.$$

Example 5 *To determine* $\displaystyle\int\frac{1}{2-2\sin x+\cos x}dx$ *using the substitution* $t=\tan\frac{x}{2}$.

Let $t=\tan\frac{x}{2}$. *Then, as shown above,*

$$\sin x=\frac{2t}{1+t^2},\qquad \cos x=\frac{1-t^2}{1+t^2}\qquad \textit{and}\qquad dx=\left(\frac{2}{1+t^2}\right)dt.$$

Therefore the integral becomes

$$\int\frac{1}{2-2\sin x+\cos x}dx=\int\left(\frac{1}{2-\frac{4t}{1+t^2}+\frac{1-t^2}{1+t^2}}\right)\left(\frac{2}{1+t^2}\right)dt$$

The expression $\left(\frac{1}{2-\frac{4t}{1+t^2}+\frac{1-t^2}{1+t^2}}\right)\left(\frac{2}{1+t^2}\right)$ *simplifies to* $\frac{2}{t^2-4t+3}=\frac{2}{(t-1)(t-3)}$. *We may split this into partial fractions and the result is*

$$\frac{2}{(t-1)(t-3)}=\frac{1}{t-3}-\frac{1}{t-1}.$$

Therefore

$$\begin{aligned}\int\frac{1}{2-2\sin x+\cos x}dx &= \int\left(\frac{1}{t-3}-\frac{1}{t-1}\right)dt=\ln|t-3|-\ln|t-1| \\ &= \ln\left|\tan\tfrac{x}{2}-3\right|-\ln\left|\tan\tfrac{x}{2}-1\right|+c\end{aligned}$$

Say we have to evaluate $\int_0^{\pi/3}\frac{1}{2-2\sin x+\cos x}dx$. *We would note that when* $x=0$, $t=0$ *and when* $x=\frac{\pi}{3}$, $t=\tan(\frac{\pi}{6})=\frac{1}{\sqrt{3}}$. *Therefore*

$$\int_0^{\pi/3}\frac{1}{2-2\sin x+\cos x}dx=\left[\ln\left|\frac{t-3}{t-1}\right|\right]_0^{1/\sqrt{3}}=\ln\left|\frac{\sqrt{3}-3}{\sqrt{3}-1}\right|-\ln 3$$

The expression $\left|\frac{\sqrt{3}-3}{\sqrt{3}-1}\right|$ *may be simplified to* $4+\sqrt{3}$, *(rationalize the denominator then cancel common factors). Then the above result may be reduced to* $\ln\left(\frac{4+\sqrt{3}}{3}\right)$.

Exercise 6

1. *Use the substitution* $t=\tan\frac{x}{2}$ *to show that* $\int_0^{\pi/2}\frac{1}{1+\sin x+\cos x}dx=\ln 2$.

2. *Use the substitution* $t=\tan\frac{x}{2}$ *to determine:*

   (a) $\int\frac{1}{1+\cos x}dx$

   (b) $\int_{\pi/3}^{\pi/2}\frac{1}{1+\sin x}dx$

   (c) $\int\frac{\sin x}{(1+\sin x)^2}dx$

   (d) $\int\frac{\cos x}{(1+\sin x)^2}dx$