Factoring a Difference of Two Squares

A difference of two squares consists of a *squared term* **minus** *another squared term.* Examples:

$$x^2 - 7^2 \qquad\qquad 3^2 - (4y)^2 \qquad\qquad \left(5a^2\right)^2 - \left(b^3\right)^2$$

$$(x+2)^2 - (3x-1)^2 \qquad\qquad (xy)^2 - \left(a^2 b\right)^2 \qquad\qquad \left(\frac{x}{3}\right)^2 - (4y)^2$$

- An expression like $4z^6 - 81x^4$ may not look like a difference of two squares, but it is. Simply write it as

$$\left(2z^3\right)^2 - \left(9x^2\right)^2$$

- Another one is $x^6 - y^6$. It becomes a difference of two squares when we write it as $\left(x^3\right)^2 - \left(y^3\right)^2$.

The technique for factoring a difference of two squares follows from the following observations:
If a and b are any numbers or algebraic expressions then

$$(a-b)(a+b) = a^2 - b^2$$

This is easily verified by removing parentheses:

$$
\begin{aligned}
(a-b)(a+b) &= a^2 + ab - ba - b^2 \\
&= a^2 - b^2
\end{aligned}
$$

Given $a^2 - b^2$, we simply work our way backwards to get

$$
\begin{aligned}
a^2 - b^2 &= a^2 - ab + ab - b^2 \\
&= a(a-b) + b(a-b) \\
&= (a-b)(a+b)
\end{aligned}
$$

Of course you do not have to repeat these steps to factor a given difference of two squares. Use the identity $a^2 - b^2 = (a-b)(a+b)$ directly. For example:

- $x^2 - 7^2 = (x-7)(x+7)$

- $3^2 - (4y)^2 = (3-4y)(3+4y)$

- $\left(5a^2\right)^2 - \left(b^3\right)^2 = \left(5a^2 - b^3\right)\left(5a^2 + b^3\right)$

- $(x+2)^2 - (3x-1)^2 = (x+2+3x-1)(x+2-(3x-1))$. This may be simplified to $(x+2)^2 - (3x-1)^2 = (4x+1)(-2x+3)$

A sum of two squares cannot be factored. For example, $x^2 + 4^2$ cannot be factored, nor can $(3x)^2 + (5y)^2$. They are prime.

Exercise 1

1. *Factor the following:*

    (a) $x^2 - 25$

    (b) $4x^2 - 49$

    (c) $x^4 - y^4$

    (d) $\dfrac{1}{x^2} - 4$

    (e) $\dfrac{x^2}{4} - \dfrac{9}{y^2}$

    (f) $9 - 49x^4$

    (g) $x^4 - y^4$

    (h) $1 - 16x^2 y^4$

    (i) $x^2 y^2 - z^4 w^6$

    (j) $\dfrac{y^2}{x^2} - 1$

    (k) $x^8 - y^8$

    (l) $\left(x^2 + 4\right)^2 - \left(x^2 - 4\right)^2$

2. *Solve each equation:*

    (a) $x^2 + 7x + 12 = 0$

    (b) $3x^2 + 5x + 20 = 2x^2 - 3x + 4$

    (c) $\sqrt{x+6} = x - 4$ *(hint: square both sides).*

3. *Factor* $4x^3 - 12x^2 - 9x + 27$ *by grouping then solve the equation*

$$4x^3 - 12x^2 - 9x + 27 = 0$$

Factoring a Difference/Sum of Two Cubes

A difference of two cubes consists of a *cubed term* **minus** *another cubed term.* Examples:

$$x^3 - 2^3 = x^3 - 8 \qquad\qquad 4^3 - (3z)^3 \qquad\qquad (4v)^3 - (5w)^3$$

The technique for factoring a difference of two squares follows from the following observations: If a and b are any numbers or algebraic expressions then

$$(a - b)\left(a^2 + ab + b^2\right) = a^3 - b^3$$

To verify it, simply remove parentheses:

$$(a - b)\left(a^2 + ab + b^2\right) \quad = \quad a^3 + a^2 b + ab^2 - ba^2 - b^2 a - b^3 = a^3 - b^3$$

We will use this result in the reverse form:

$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

Applying it to the above differences of cubes, we get

- $x^3 - 2^3 = (x - 2)\left(x^2 + 2x + 4\right)$

- $4^3 - (3z)^3 = (4 - 3z)\left(4^2 + 4\,(3z) + (3z)^2\right) = (4 - 3z)\left(16 + 12z + 9z^2\right)$

- $(4v)^3 - (5w)^3 = (4v - 5w)\left(16v^2 + 20vw + 25w^2\right)$

A sum of two cubes consists of a *cubed term* **plus** *another cubed term.* Examples:

$$y^3 + 4^3 = y^3 + 64 \qquad\qquad x^3 + (4y)^3 \qquad\qquad (2d)^3 + (3c)^3$$

Unlike a sum of two squares, a sum of two cubes can be factored. This follows from the identity

$$(a + b)\left(a^2 - ab + b^2\right) = a^3 + b^3$$

To verify it, simply remove parentheses:

$$(a + b)\left(a^2 - ab + b^2\right) = a^3 - a^2 b + ab^2 + ba^2 - b^2 a + b^3 = a^3 + b^3$$

We will use it in the reverse form:

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right) \tag{1}$$

Note the positive sign in $(a + b)$ and the negative sign in $\left(a^2 - ab + b^2\right)$.

Applying (1) to the above sums of cubes, we get

- $y^3 + 4^3 = (y + 4)\left(y^2 - 4y + 16\right)$

- $x^3 + (4y)^3 = (x + 4y)\left(x^2 - x\,(4y) + (4y)^2\right) = (x + 4y)\left(x^2 - 4yx + 16y^2\right)$

- $(2d)^3 + (3c)^3 = (2d + 3c)\left(4d^2 - 6dc + 9c^2\right)$

Exercise 2 *Factor the given sum or difference of cubes*

$$\text{(a) } x^3 + 4^3 \qquad \text{(b) } x^3 - 4^3 \qquad \text{(c) } (2x)^3 - (5y)^3 \qquad \text{(d) } \frac{y^3}{64} - \frac{64}{y^3}$$

$$\text{(e) } \left(3b^2\right)^3 + \left(2a^2\right)^3 \quad \text{(f) } 1 - \frac{1}{b^3} \qquad \text{(g) } 27a^6 + 8b^{12} \qquad \text{(h)}1 - \frac{8}{x^3}$$

$$\text{(i) } \frac{a^3}{27} + \frac{8}{b^6} \qquad\qquad \text{(j) } 64x^6 - 125y^3 \quad \text{(k) } 64x^9 - 27y^{12} \quad \text{(l) } a^6 b^3 - 27x^9$$

A Useful Identity We Must Mention

We saw that $a^3 - b^3$ may be factored as

$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right) \tag{2}$$

Substituting $a = 1$ and $b = x$ in (2) gives

$$1 - x^3 = (1 - x)\left(1 + x + x^2\right)$$

It turns out that $1 - x^4 = (1 - x)\left(1 + x + x^2 + x^3\right)$. Confirm by multiplying. Use the pattern suggested by these two cases to complete the following:

$$
\begin{aligned}
1 - x^5 &= (1 - x)\,(\qquad\qquad\qquad) \\
1 - x^6 &= (1 - x)\,(\qquad\qquad\qquad) \\
1 - x^n &= (1 - x)\,(\qquad\qquad\qquad\qquad\qquad)\ \text{where } n \text{ is a positive integer.}
\end{aligned}
$$

We will use these expressions in a modified form. For example, we may use $(1 - x)\left(1 + x + x^2\right) = 1 - x^3$ to write

$$1 + x + x^2 = \frac{1 - x^3}{1 - x} = \frac{x^3 - 1}{x - 1}$$

The usefulness of this expression is that one replaces a sum of terms 1, x and x^2 with a quotient of two simple expressions.

Likewise, we may use $1 - x^4 = (1 - x)\left(1 + x + x^2 + x^3\right)$ to write

$$1 + x + x^2 + x^3 = \frac{x^4 - 1}{x - 1}$$

In this case we have replaced a sum of the four terms 1, x, x^2 and x^3 with a quotient of two simple expressions.

Complete the following, (n is a positive integer):

$$
\begin{aligned}
1 + x + x^2 + x^3 &= \underline{\qquad\qquad} \\
1 + x + x^2 + x^3 + x^4 &= \underline{\qquad\qquad} \\
1 + x + x^2 + x^3 + x^4 + x^5 &= \underline{\qquad\qquad} \\
1 + x + x^2 + x^3 + x^4 + \cdots + x^n &= \underline{\qquad\qquad}
\end{aligned}
$$

Exercise 3 *Use the formula*

$$1 + x + x^2 + x^3 + x^4 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x} = \frac{x^{k+1} - 1}{x - 1} \tag{3}$$

to calculate each sum:

1. $1 + 3 + 3^2 + 3^3 + 3^4$

2. $1 + (-5) + (-5)^2 + (-5)^3 + (-5)^4 + (-5)^5 + (-5)^6$

3. $2 + 2^2 + 2^3 + 2^4 + 2^5 = 2(1 + 2 + 2^2 + 2^3 + 2^4)$

4. $\left(-\dfrac{2}{3}\right) + \left(-\dfrac{2}{3}\right)^2 + \left(-\dfrac{2}{3}\right)^3 + \cdots + \left(-\dfrac{2}{3}\right)^{20}$

5. $2000\left[1 + (1.02) + (1.02)^2 + (1.02)^3 + \cdots + (1.02)^{20}\right]$