More General Limits

In order to prove a number of results we use in this course, one must have a precise definition of a limit. So far we have considered the limit as h approaches 0 of the special quotient

$$\frac{f(x+h)-f(x)}{h}.$$

To determine it, you had to answer the question: *what single number is close to all the values you get when you substitute numbers h close to 0 into the expression $\frac{f(x+h)-f(x)}{h}$?* The answer to this question is the derivative of f at x, denoted by $f'(x)$. Now we are going to ask the following more general question:

> *Let g be a given function and c be a given real number. Suppose you choose different numbers x **close**, (but not equal) to c and determine the corresponding values $g(x)$. Is there a single number l that is **close** to all such values?*

If the answer is YES, then l is called the limit of $g(x)$ as x approaches c. This intuitive definition of a limit suffices to introduce derivatives of the trigonometric and the exponential functions. The precise definition is addressed later on.

Example 1 *Consider the function g with formula $g(x)=\frac{6x+6}{x^3+1}$, $x\neq -1$. Although $g(-1)$ is not defined, we can evaluate $g(x)$ for all the numbers x "close" to -1. The table below gives a sample of such values.*

x	-1.5	-1.1	-1.002	-1.0001	-0.9998	-0.999	-0.9	-0.56
$\frac{6x+6}{x^3+1}$	1.26	1.81	1.996	1.9998	2.0004	2.002	2.21	3.202

It appears that every number x that is sufficiently close to -1 gives a value $g(x)$ close to 2, therefore $g(x)$ should have limit 2 as x approaches -1. We can actually defend this claim more rigorously as follows: A number x close to -1 has the form $x=-1+h$ where h is close to 0. (The smaller h is , the closer $x=-1+h$ is close to -1.) It gives a value

$$\begin{aligned} g(x) &= g(-1+h)=\frac{6(-1+h)+6}{(-1+h)^3+1}=\frac{-6+6h+6}{-1+3h-3h^2+h^3+1} \\ &= \frac{6h}{3h-3h^2+h^3}=\frac{6}{3-3h+h^2} \end{aligned}$$

When h is close to 0, (which means that $x=-1+h$ is close to -1), then $\frac{6}{3-3h+h^2}$ is close to $\frac{6}{3-0+0}=$ 2.

Example 2 *Let $g(x)=\frac{x^2-1}{\sqrt{x}-1}$, $x\neq 1$. Then $g(x)$ has limit 4 as x approaches 1. (Substitute a couple of numbers x close to 1 into $\frac{x^2-1}{\sqrt{x}-1}$.) To verify it, note that a number x close to 1 has the form $x=1+h$ where h is close to 0. It gives a value*

$$g(1+h)=\frac{(1+h)^2-1}{\sqrt{1+h}-1}=\frac{2h+h^2}{\sqrt{1+h}-1}$$

You should anticipate rationalizing the denominator to get

$$\begin{aligned} f(1+h) &= \frac{h(2+h)\left(\sqrt{1+h}+1\right)}{\left(\sqrt{1+h}-1\right)\left(\sqrt{1+h}+1\right)}=\frac{h(2+h)\left(\sqrt{1+h}+1\right)}{1+h-1} \\ &= (2+h)\left(\sqrt{1+h}+1\right). \end{aligned}$$

Clearly, $(2+h)\left(\sqrt{1+h}+1\right)$ *is close to* $(2+0)\left(\sqrt{1+0}+1\right)=4$ *when* h *is close to 0, hence* $g(x)$ *has limit 4 as* x *approaches 1 and we may write* $\lim_{x\to 1}\frac{x^2-1}{\sqrt{x}-1}=4$.

Exercise 3

1. *Let* $f(x)=\frac{3-x}{3-\sqrt{x+6}}$. *Verify that when* x *is close to 3 then* $f(x)$ *is close to* 6.

2. *Set your calculator in radian mode then use it to complete the following table*

h	-1	-0.05	-0.05	-0.006	-0.00005	0.00004	0.008	0.02	0.6	0.9
$\sin h$										
$1-\cos h$										
$\frac{\sin h}{h}$										
$\frac{1-\cos h}{h}$										
$\frac{1-\cos h}{h^2}$										

   (a) *What value does the table suggest for* $\lim_{h\to 0}\frac{\sin h}{h}$*?*

   (b) *What value does the table suggest for* $\lim_{h\to 0}\frac{1-\cos h}{h}$*?*

   (c) *What value does the table suggest for* $\lim_{h\to 0}\frac{1-\cos h}{h^2}$*?*

3. *Use a calculator to complete the following table then guess* $\lim_{h\to 0}\frac{e^h-1}{h}$.

h	-0.1	-0.02	-0.007	-0.001	-0.0002	0.00001	0.001	0.008	0.06	0.3
$\frac{e^h-1}{h}$										