# Integration by Parts

This is essentially the reverse of the product rule for derivatives. Recall that the derivative of a product $f(x)g(x)$ is

$$\frac{d}{dx}(f(x)g(x)) = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx}.$$

We may re-arrange this as

$$g(x)\frac{df}{dx} = \frac{d}{dx}(f(x)g(x)) - f(x)\frac{dg}{dx} \tag{1}$$

Now consider antiderivatives of the terms in (1).

- An antiderivative of $g(x)\frac{df}{dx}$ is written as $\int g(x)\frac{df}{dx}dx$.
- An obvious antiderivative of $\frac{d}{dx}(f(x)g(x))$ is $f(x)g(x)$.
- An antiderivative of $-f(x)\frac{dg}{dx}$ is written as $-\int f(x)\frac{dg}{dx}dx$.

Therefore

$$\int g(x)\frac{df}{dx}dx = f(x)g(x) - \int f(x)\frac{dg}{dx}dx \tag{2}$$

This is the formula for integrating by parts. To apply it to an integral $\int h(x)dx$, do the following:

(a) Write $h(x)$ as a product of two functions. Call one of them $g(x)$ and the other one $\frac{df}{dx}$.

(b) Determine $f(x)$ from your knowledge of $\frac{df}{dx}$ and differentiate $g$ to get $\frac{dg}{dx}$.

(c) Substitute $f(x)$, $g(x)$, $\frac{df}{dx}$ and $\frac{dg}{dx}$ into (2).

Your choice of $f(x)$ and $g(x)$ should be such that $\int f(x)\frac{dg}{dx}dx$ is easier to evaluate than $\int g(x)\frac{df}{dx}dx$.

**Example 1** *To determine* $\int x \sin x dx$.

*We choose* $g(x) = x$ *and* $\frac{df}{dx} = \sin x$. *Then* $f(x) = -\cos x$, *(there is no need to introduce a constant of integration at this stage), and* $\frac{dg}{dx} = 1$. *Substituting these into (2) gives*

$$\int x \sin x dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + c$$

Note that the choice $g(x) = \sin x$ and $\frac{df}{dx} = x$ leads to

$$\int x \sin x dx = \tfrac{1}{2}x^2 \sin x - \int \tfrac{1}{2}x^2 \cos x dx$$

which is no easier to evaluate than $\int x \sin x dx$.

It may be necessary to apply the technique more than once as the next example shows:

**Example 2** *To determine* $\int x^2 e^{2x} dx$, *let* $g(x) = x^2$ *and* $\frac{df}{dx} = e^{2x}$. *Then* $\frac{dg}{dx} = 2x$ *and* $f(x) = \frac{1}{2}e^{2x}$. *Substituting into (2) gives*

$$\int x^2 e^{2x} dx = \tfrac{1}{2}x^2 e^{2x} - \int x e^{2x} dx.$$

*We have reduced the power of $x$ under the integral sign by 1. To evaluate $\int xe^{2x}dx$, we integrate by parts a second time. Take $g(x) = x$ and $\frac{df}{dx} = e^{2x}$. (Hopefully, you do not confuse the $g(x)$ in this part of the solution with the $g(x)$ in the first part.) Then $\frac{dg}{dx} = 1$ and $f(x) = \frac{1}{2}e^{2x}$. Substituting into (2) gives*

$$\int xe^{2x}dx = \tfrac{1}{2}xe^{2x} - \int \tfrac{1}{2}e^{2x}dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + c$$

*Therefore* $\int x^2e^{2x}dx = \frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right) + c = \frac{1}{2}e^{2x}\left(x^2 - x + \frac{1}{2}\right) + c.$

In the next example, we write the integrand $h(x)$ as $h(x) \cdot 1$ then choose $g(x) = h(x)$ and $\frac{df}{dx} = 1$.

**Example 3** *To determine $\int \arcsin x dx$, choose $g(x) = \arcsin x$ and $\frac{df}{dx} = 1$. Then $f(x) = x$ and $\frac{dg}{dx} = \frac{1}{\sqrt{1-x^2}}$. Therefore*

$$\int \arcsin x dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx.$$

*You should be able to determine $\int \frac{x}{\sqrt{1-x^2}}dx = \int x\left(1-x^2\right)^{-1/2} dx$ by inspection. The result is*

$$\int \arcsin x dx = x \arcsin x + \sqrt{1-x^2} + c$$

**Exercise 4**

1. *Integrate $\int x \cos x dx$ by parts.*
2. *Integrate $\int x^2 \ln x dx$ by parts.*
3. *Show that if $k \neq -1$ and it is a constant then $\int x^k \ln x dx = \frac{x^{k+1}}{k+1}\left(\ln x - \frac{1}{k+1}\right)$.*
4. *Use the substitution $u = \ln x$ to show that $\int \frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2 + c$.*
5. *Use the result of Exercises 3 to determine $\int \ln x dx$.*
6. *Determine $\int \arctan x dx$.*
7. *Determine $\int x \arctan x dx$. (Hint: $\frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2} = 1 - \frac{1}{1+x^2}$.)*
8. *Integrate $\int \frac{x^3}{\sqrt{x^2+1}}dx$ by parts. (Hint: Take $g(x) = x^2$ and $\frac{df}{dx} = \frac{x}{\sqrt{x^2+1}}$. You should be able to determine $f$ by inspection.)*

9. *In some instances, two or more integrations by parts may lead one to an integral involving the given integrand . One is then able to form an equation involving the given integral and solve for it. For an example, consider*

$$\int e^{2x} \cos x dx.$$

*Take* $g(x) = e^{2x}$ *and* $\frac{df}{dx} = \cos x$. *Then* $\frac{dg}{dx} = 2e^{2x}$, $f(x) = \sin x$ *and*

$$\int e^{2x} \cos x dx = e^{2x} \sin x - 2 \int e^{2x} \sin x dx \tag{3}$$

*The next step is to integrate* $\int e^{2x} \sin x dx$ *by parts. Take* $g(x) = e^{2x}$ *and* $\frac{df}{dx} = \sin x$. *Then* $\frac{dg}{dx} = 2e^{2x}$ *and* $f(x) = -\cos x$, *therefore*

$$\int e^{2x} \sin x dx = -e^{2x} \cos x + \int 2e^{2x} \cos x dx. \text{ Substituting into(3) gives}$$

$$\int e^{2x} \cos x dx = e^{2x} \sin x + 2e^{2x} \cos x - 4 \int e^{2x} \cos x dx. \quad \text{In other words,}$$

$$\int e^{2x} \cos x dx = e^{2x} \sin x + 2e^{2x} \cos x - 4 \int e^{2x} \cos x dx \tag{4}$$

*Note that the given integral* $\int e^{2x} \cos x dx$ *appears in the right hand side. We now solve (4) for the integral and the result is*

$$\int e^{2x} \cos x dx = \frac{e^{2x} (\sin x + 2 \cos x)}{5} + c$$

*Let* $a$ *and* $b$ *be constants. Use a similar procedure to:*

(a) *Determine* $\int e^{2x} \sin 3x dx$.

(b) *Show that* $\int e^{ax} \sin bx dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2} + c$

(c) *Show that* $\int e^{ax} \cos bx dx = \frac{e^{ax} (b \sin bx + a \cos bx)}{a^2 + b^2} + c$

10. *Let* $n$ *and* $b$ *be constants. Show that*

$$\int x^n e^{bx} dx = \frac{x^n e^{bx}}{b} - \frac{n}{b} \int x^{n-1} e^{bx} dx. \tag{5}$$

11. *Formula (5) is an example of a reduction formula. Using it enables one to reduce the exponent of* $x$, *hence the term "reduction formula". Let us apply it repeatedly to determine* $\int x^3 e^{4x} dx$ . *Thus*

$$\int x^3 e^{4x} dx = \frac{x^3 e^{4x}}{4} - \tfrac{3}{4} \int x^2 e^{4x} dx, \qquad \text{(first application)}$$

*The second step is to apply it to* $\int x^2 e^{4x} dx$ *and the result is*

$$\int x^2 e^{4x} dx = \frac{x^2 e^{4x}}{4} - \tfrac{1}{2} \int x e^{4x} dx.$$

*Therefore*

$$\int x^3 e^{4x} dx = \frac{x^3 e^{4x}}{4} - \tfrac{3}{4} \left( \frac{x^2 e^{4x}}{4} - \tfrac{2}{4} \int x e^{4x} dx \right) = \left( \frac{x^3}{4} - \frac{3x^2}{16} \right) e^{4x} + \tfrac{3}{8} \int x e^{4x} dx$$

*Finally, we apply it to* $\int xe^{4x}dx$ *to get*

$$\int xe^{4x}dx = \frac{xe^{4x}}{4} - \frac{1}{4}\int e^{4x}dx = \frac{xe^{4x}}{4} - \frac{e^{4x}}{16} + c$$

*The last integral was determined by inspection. Therefore*

$$\int x^3 e^{4x}dx = \left(\frac{x^3}{4} - \frac{3x^2}{16} + \frac{3x}{32} - \frac{3}{128}\right)e^{4x} + c$$

*Determine* $\int x^4 e^{2x}dx$ *in a similar way.*

12. *Use your answer to question 9c above to determine* $\int e^{5x}\cos 6x dx$.

13. *Use the substitution* $x = u^2$ *to show that* $\int \cos\sqrt{x}dx = 2\int u\cos u du$. *Now integrate by parts to complete the integration.*

14. *Consider the integral* $\int \cos^n x dx$ *where* $n$ *is a non-zero constant. We may write* $\cos^n x$ *as* $\left(\cos^{n-1}x\right)(\cos x)$. *Now take* $g(x) = \cos^{n-1}x$ *and* $\frac{df}{dx} = \cos x$ *and integrate by parts to get*

$$\int \cos^n x dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\sin^2 x dx$$

*Use the identity* $\sin^2 x = 1 - \cos^2 x$ *to deduce that*

$$\int \cos^n x dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x dx \qquad (6)$$

*This is another reduction formula which we may use to integrate powers of* $\cos x$. *For example,*

$$\int \cos^3 x dx = \frac{\cos^2 x\sin x}{3} + \frac{2}{3}\int \cos x dx = \frac{\cos^2 x\sin x}{3} + \frac{2\sin x}{3} + c$$

15. *Let* $n$ *be a non-zero constant. Show that*

$$\int \sin^n x dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x dx \qquad (7)$$

*then use this reduction formula to determine* $\int \sin^3 x dx$.

16. *Assume that* $n$ *is a constant that is not equal to 1. Complete the following exercise to derive the following reduction formula:*

$$\int \tan^n x dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x dx \qquad (8)$$

*We write* $\tan^n x$ *as* $\tan^{n-2}x\tan^2 x$. *Then*

$$\begin{aligned}\int \tan^n x dx &= \int \tan^{n-2}x\tan^2 x dx = \int \tan^{n-2}x(\sec^2 x - 1)dx \\ &= \int \tan^{n-2}x\sec^2 x dx - \int \tan^{n-2}x dx\end{aligned}$$

*Now integrate* $\int \tan^{n-2}x\sec^2 x dx$ *by inspection, (if you can't make a substitution* $u = \tan x$*), and complete the exercise.*

*Use the reduction formula to determine* $\int \tan^3 x dx$. *(Remember that* $\int \tan x dx = \ln|\sec x| + c$.*)*

17. *Assume that n is a constant that is not equal to 1. Complete the following exercise to derive the following reduction formula:*

$$\int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x dx. \tag{9}$$

*We write* $\sec^n x$ *as* $\sec^{n-2} x \sec^2 x$. *Then* $\int \sec^n x dx = \int \sec^{n-2} x \sec^2 x dx$. *Now integrate by parts.*

*Use the reduction formula to determine* $\int \sec^4 x dx$.

18. *Show that if* $n \neq -1$ *then*

$$\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx$$

19. *Show that if* $n \neq -1$ *then*

$$\int \csc^n x dx = -\frac{\csc^{n-1} x \cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x dx$$

*Use the formula to determine* $\int \csc^3 x dx$. *(Remember that* $\int \csc x dx = -\ln|\csc x + \cot x| + c$*)*