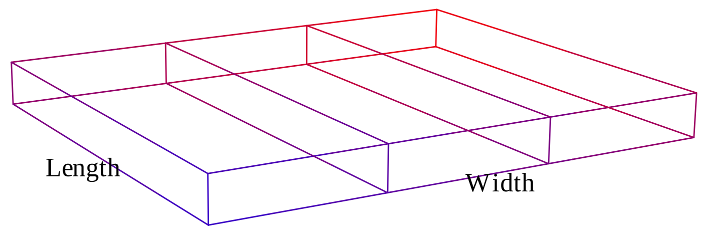
Two More Examples Followed by Group Projects

Example 1 *A farmer has 500 yards of fencing to construct three equal enclosures, shown below (not drawn to scale), for his animals.*



What is the largest possible area he can enclose?

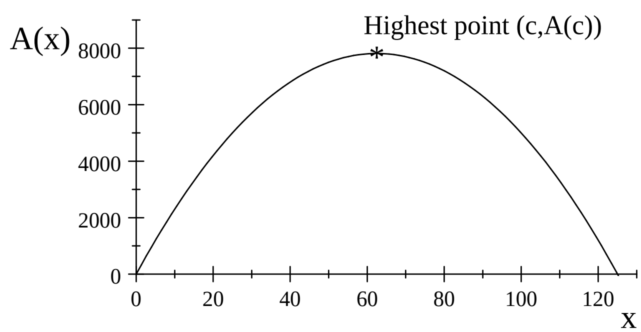
The following table may shed light on how to solve this problem.

Length of enclosure	10	30	45	60	70	80	95	120
Width of enclosure	230	190	160	130	110	90	60	10
Total enclosed area	2300	5700	7200	7800	7700	7200	5700	1200

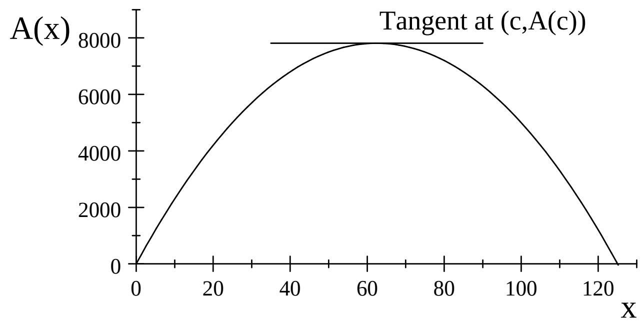
The table shows that the area increases as the length increases, until it reaches a maximum value when the length is equal to some number c between 45 and 70 yards. Increasing the length beyond c yard leads to a reduction in the enclosed area. To determine c we need an expression that gives the enclosed area in terms of the length of the enclosure. Suppose the length is x yards. Then $4x$ yards are needed to construct the two inner dividers and the two sides that are parallel to the dividers. That leaves $500 - 4x$ yards for the other two sides. It follows that the enclosure is $\frac{1}{2}(500 - 4x)$ yards wide. This implies that the total enclosed area, (in square yards), is

$$A(x) = (x)\,\frac{1}{2}(500 - 4x) = 250x - 2x^2$$

A graph of A is shown below



We want the value of c that gives the largest possible value of the area A. In the figure above, it is the number c corresponding to the highest point on the graph of A. Once again, c is characterized by the fact that the tangent to the graph of A at $(c, A(c))$ is horizontal, and so $A'(c) = 0$.



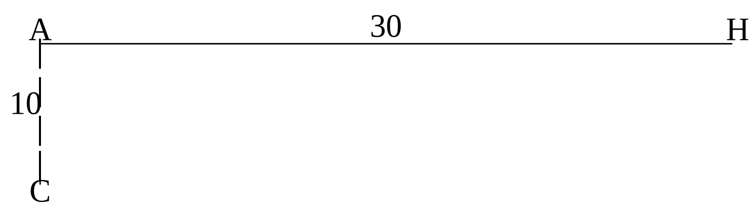
Therefore we need the solution to the equation $A'(x) = 0$. Since $A'(x) = 250 - 4x$ we must solve the equation

$$A'(x) = 250 - 4x$$

The solution is $x = \frac{250}{4}$, hence the largest possible area that may be enclosed is

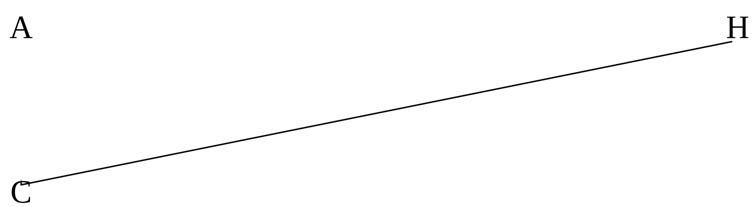
$$250\left(\frac{250}{4}\right) - 2\left(\frac{250}{4}\right)^2 = 7812.5 \text{ square yards.}$$

Example 2 *An accident patient has to be rushed by an emergency vehicle from a camp in a desert to a hospital. The camp C is 10 miles from a highway, (the distance from A to C), and the hospital H is 30 miles down the highway, (the distance from A to H), as shown in the figure below.*

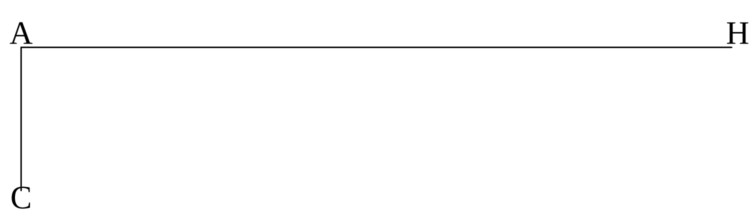


The emergency vehicle can be driven through the desert sand at 50 miles per hour, and along the highway at 90 miles per hour. What is the shortest possible time to get the patient to hospital? (In a situation like this, minutes and seconds matter.)

The driver of the emergency vehicle has a number of options. One of them is to drive directly to the hospital through the desert along the path shown below. This is a $\sqrt{1000}$ miles trip and it takes $\sqrt{1000}/50$ hours, which is close to 38 minutes.



Another option is to drive 10 miles directly to the highway and then drive the 30 miles along the highway.



This takes $10/50 + 30/90$ hours, or 32 minutes. There is also an option of driving through the desert to some point B between A and H as shown below, then drive the remaining miles on the highway.

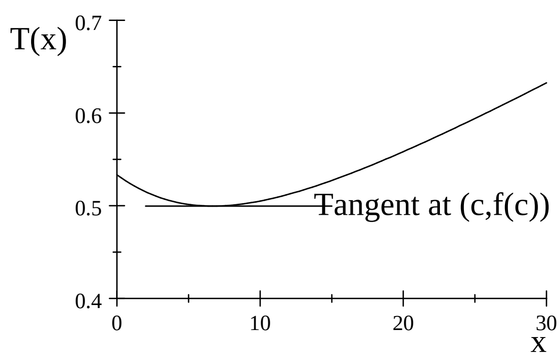
The following table gives the times for a sample of such points.

Value of x	0	2	4	6	8	10	20
Total time for trip (in hours to 3 dec. pl.).	0.533	0.515	0.504	0.500	0.501	0.505	0.558

To handle the general case, suppose he drives through the desert to a point, on the highway, that is x miles from A, then drives the remaining $30 - x$ miles on the highway. This means that he drives $\sqrt{100 + x^2}$ miles through the desert. At 50 miles per hour, it takes him $\left(\sqrt{100 + x^2}\right)/50$ hours. The remaining $30 - x$ miles on the highway take him $(30 - x)/90$ hours. Therefore, the total time, in hours, for the trip is

$$T(x) = \frac{\sqrt{100 + x^2}}{50} + \frac{30 - x}{90} = \frac{1}{50}\left(100 + x^2\right)^{1/2} + \frac{1}{3} - \frac{x}{90}$$

Clearly $0 \leq x \leq 30$. The graph of T is given below.



We have to find the distance c corresponding to the lowest point $(c, T(c))$ on the graph of T. Then $T(c)$ is the shortest possible time. As expected, c is characterized by the fact that the tangent to the graph of T at $(c, T(c))$ is horizontal, so that $T'(c) = 0$. Therefore, it is the solution of the equation

$$0 = T'(x) = \frac{1}{50}\left(\frac{1}{2}\right)\left(100 + x^2\right)^{-1/2}(2x) - \frac{1}{90} = \frac{x}{50\sqrt{100 + x^2}} - \frac{1}{90}$$

that is between 0 and 30. Clearing fractions and radicals gives

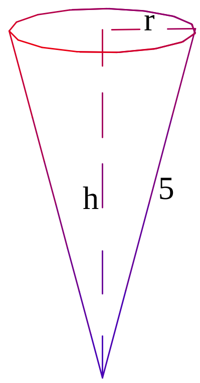
$$25\left(100 + x^2\right) = 81x^2$$

The positive solution of this equation is $x = \sqrt{2500/56} = 6.68$ (to 2 decimal places). Therefore the shortest possible time is $T(\sqrt{2500/56}) = 0.4996$ hours (to 4 decimal places). This is just a little less than 30 minutes.

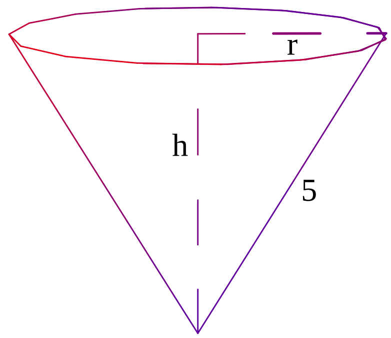
Group Projects

1. Suppose the farmer in the above example wants to construct two enclosures instead of three. What is the largest possible area he can enclose with the 500 yards of fencing?

2. You have to make an ice-cream cone, (more precisely, a right circular cone), with a slant length of 5 inches. (See the figures below for the definition of the slant length of a right circular cone.)



A tall and thin cone



A wider and shorter cone

You could make it tall and thin or wide but shorter as shown in the figures above. Let h and r, (in inches), be its height and radius respectively, and V, in cubic inches be its volume. These variables are related by the equation $V = \frac{1}{3}\pi r^2 h$. Copy and complete the following table.

h	0.5	1	1.5	2	2.5	4	4.5	h
r								
V								

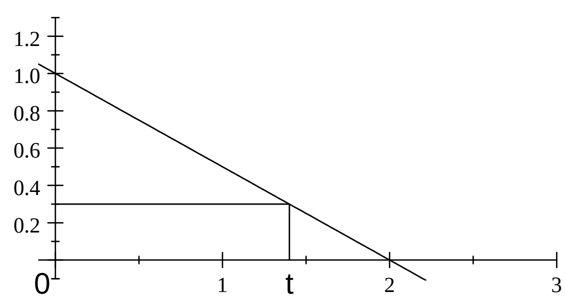
Use the above information to determine the largest volume of a right circular cone with a slant length of 5 inches.

3. Consider the patient, in the above example being rushed to hospital.

   (a) Determine c when the speed through the desert is v miles per hour. (Your answer should depend on v.)

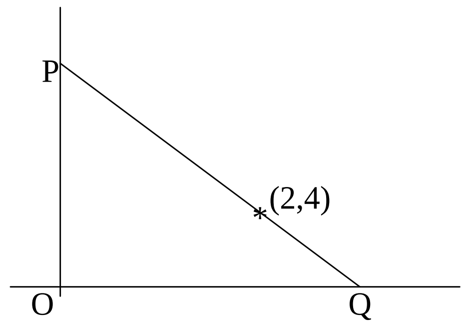
   (b) When the speed through the desert is higher than some value u, the shortest time is attained by driving directly through the sand to the hospital. Use your result in part (a) to determine u.

4. In the figure below, a rectangle is drawn inside the region in the first quadrant enclosed by the two coordinate axes and the line $y = \frac{1}{2}(2 - x)$. The vertical sides of the rectangle are on the lines $x = 0$ and $x = t$.



Show that the area of the rectangle is $t - \frac{1}{2}t^2$ then the largest possible area of such a rectangle

5. Consider a line segment in the **first quadrant** satisfying the following conditions: (i) it passes through $(2,4)$, (ii) it intersects the y-axis at a point P, (iii) it intersects the x-axis at a point Q.



Show that when the slope of PQ is m then the coordinates of P and Q are $(0, 4-2m)$ and $\left(2-\frac{4}{m}, 0\right)$ respectively. Use this information to calculate the area of triangle PQO in terms of m then determine the minimum area of such a triangle.

6. You are required to design a cylindrical can, without a lid, that must contain 22 cubic centimeters of fluid and uses the least amount of material for construction. It is necessary to express the area of the material used to construct it in terms of some variable. Say you choose to express it in terms of the radius r of the can. Show that the height of the can must be $h = \frac{22}{\pi r^2}$ cm and the area of the material needed to construct it is $A = \pi r^2 + \frac{44}{r}$ square centimeters. Use this information to determine the radius r that corresponds to the smallest possible area then determine the dimensions of the can with the required property.

7. Consider the straight line $y = 2x+1$. A general point on the line has coordinates $(x, 2x+1)$. Show that its distance from $(2,-2)$ is $\sqrt{5x^2+8x+13}$ then use an argument involving derivatives to find the shortest distance from the line to the point $(2,-2)$.

8. This generalizes the result of the above exercise: Consider the straight line $y = ax+b$ and a point (u,v) in the plane. A general point on the given line has coordinates $(x, ax+b)$. Show that if D is its the distance from (u,v) then

$$D^2 = (x-u)^2 + (ax+b-v)^2 .$$

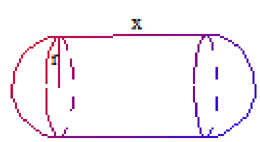
Use an argument involving derivatives to show that the point on the line that is closest to (u,v) has coordinates

$$\left(\frac{u+a(v-b)}{1+a^2}, \frac{b+a(u+av)}{1+a^2}\right).$$

Now deduce that the shortest distance from the point (u,v) to the line $y = ax+b$ is

$$\frac{|v-au-b|}{\sqrt{1+a^2}}$$

9. (For this problem, you need to know that a sphere with radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$; while a right cylinder with radius r and height h has volume $\pi r^2 h$ and surface $2\pi r h$.) A pharmaceutical company has to make medicine capsules in the shape of a right cylinder with radius r and length x, to which is attached hemispheres of radius r at both ends as shown in the figure below.



Each capsule must have a volume of V cubic millimeters.

(a) Show that the length x and the radius r shown in diagram are related by the equation $V = \frac{4\pi r^3}{3} + \pi r^2 x$ then and deduce that

$$x = \frac{V}{\pi r^2} - \frac{4r}{3}.$$

(b) Show that the surface area A of the capsule is

$$A = \frac{4\pi r^2}{3} + \frac{2V}{r}$$

(c) The company would like to produce capsules with the least possible surface area to minimize the cost of the coating material. Show that the radius corresponding to the smallest possible surface area is $r = \left(\frac{3V}{4\pi}\right)^{1/3}$. What is the minimum possible surface area?