The General Power Rule and the Product Rule

We know how to find the derivative of any power $u(x) = x^r$ where r is any real number. The general power rule enables us to find the derivatives of more general powers like

$$w(x) = \left(x - \frac{2}{x^2}\right)^6 \qquad h(x) = (5x - 13)^{-3} \qquad u(x) = \sqrt{x^2+1} = \left(x^2+1\right)^{1/2}$$

They all have the general form $g(x) = [f(x)]^r$ where $f(x)$, (popularly called the inner function), is some given function and r is a given number. In the case of $w(x) = \left(x - \frac{2}{x^2}\right)^6$, the inner function is $f(x) = x - \frac{2}{x^2}$ and $r = 6$.

To derive the rule, we need to know how to factor differences of squares, cubes, etc. Most probably you have already come across the formula for a difference of two squares which is

$$X^2 - Y^2 = (X - Y)(X + Y),$$

A formula for the difference of two cubes is

$$X^3 - Y^3 = (X - Y)\left(X^2 + XY + Y^2\right).$$

To verify it, simply expand the right hand side:

$$(X - Y)\left(X^2 + XY + Y^2\right) = x^3 + X^2Y + XY^2 - X^2Y - XY^2 - Y^3 = X^3 - Y^3$$

For a difference of two fourth powers, we choose

$$X^4 - Y^4 = (X - Y)\left(X^3 + X^2Y + XY^2 + Y^3\right).$$

There are other ways of factoring $X^4 - Y^4$, but we prefer this particular one. You may check that it is an identity by expanding the right hand side.

The pattern should be clear by now. Use it to complete the following; (in the last identity, n is a positive integer).

$X^5 - Y^5 = (X - Y)($

$X^6 - Y^6 = (X - Y)($

$X^7 - Y^7 = (X - Y)($

$X^n - Y^n = (X - Y)($

Now consider a function $g(x) = [f(x)]^n$ where n is a positive integer. From the definition, the derivative of g is g' with formula

$$g'(x) = \lim_{h \to 0} \frac{[f(x+h)]^n - [f(x)]^n}{h}$$

We may factor $[f(x+h)]^n - [f(x)]^n$ as

$$[f(x+h) - f(x)]\left[(f(x+h))^{n-1} + (f(x+h))^{n-2} f(x) + \cdots + (f(x))^{n-1}\right].$$

It follows that $g'(x)$ is given by

$$g'(x) = \lim_{h \to 0} \frac{[f(x+h) - f(x)]\left[(f(x+h))^{n-1} + (f(x+h))^{n-2} f(x) + \cdots + (f(x))^{n-1}\right]}{h}$$

It is convenient to write the above limit as

$$\lim_{h \to 0} \left[(f(x+h))^{n-1} + (f(x+h))^{n-2} f(x) + \cdots + (f(x))^{n-1}\right] \cdot \frac{[f(x+h) - f(x)]}{h}$$

The limit of $\dfrac{[f(x+h)-f(x)]}{h}$ as h approaches 0 is $f'(x)$. It remains to figure out the limit of

$$\left[(f(x+h))^{n-1}+(f(x+h))^{n-2}f(x)+(f(x+h))^{n-3}(f(x))^2+\cdots+(f(x))^{n-1}\right]$$

When h is close to 0, the first term $(f(x+h))^{n-1}$ in the square brackets is close to $(f(x))^{n-1}$. The second one is close to $(f(x))^{n-2}\cdot f(x)$ which also simplifies to $(f(x))^{n-1}$. The third one is also close to $(f(x))^{n-1}$. In fact, each one is close to $(f(x))^{n-1}$. Since there are n terms inside the square brackets,

$$\lim_{h\to 0}\left[(f(x+h))^{n-1}+(f(x+h))^{n-2}f(x)+\cdots+(f(x))^{n-1}\right]=n\,(f(x))^{n-1}$$

It stands to reason that when h is close to 0 then

$$\left[(f(x+h))^{n-1}+(f(x+h))^{n-2}f(x)+\cdots+(f(x))^{n-1}\right]\cdot\frac{[f(x+h)-f(x)]}{h}$$

is close to $n[f(x)]^{n-1}\cdot f'(x)$. Therefore, if n is a positive integer, then the derivative of $[f(x)]^n$ is $n[f(x)]^{n-1}\cdot f'(x)$.

Example 1

- *In the case of* $g(x)=\left(x^2+4\right)^4$, $f(x)=x^2+4$, $f'(x)=2x$ *and* $n=4$ *hence its derivative is* $g'(x)=4\left(x^2+4\right)^3(2x)=8x\left(x^2+4\right)^3$
- *In the case of* $h(x)=\left(x-\frac{2}{x^2}\right)^6$, $f(x)=x-\frac{2}{x^2}$, $f'(x)=1+\frac{4}{x^3}$ *and* $n=6$ *hence its derivative is* $h'(x)=6\left(x-\frac{2}{x^2}\right)^5\left(1+\frac{4}{x^3}\right)$.
- *In the case of* $u(x)=\left(5x^4-2x\right)^3$, $f(x)=5x^4-2x$, $f'(x)=20x^3-2$ *and* $n=3$ *hence its derivative is* $u'(x)=3\left(5x^4-2x\right)^2\left(20x^3-2\right)$
- *In the case of* $v(x)=\left(\frac{2}{x}+\frac{x}{2}\right)^5$, $f(x)=\frac{2}{x}+\frac{x}{2}$, $f'(x)=-\frac{2}{x^2}+\frac{1}{2}$ *and* $n=5$ *hence its derivative is* $v'(x)=5\left(\frac{2}{x}+\frac{x}{2}\right)^4\left(-\frac{2}{x^2}+\frac{1}{2}\right)$
- *In the case of* $w(x)=\left(4+\frac{3}{x}\right)^9$, $f(x)=4+\frac{3}{x}$, $f'(x)=-\frac{3}{x^2}$ *and* $n=9$ *hence its derivative is* $w'(x)=9\left(4+\frac{3}{x}\right)^8\left(-\frac{3}{x^2}\right)=-\frac{27}{x^2}\left(4+\frac{3}{x}\right)^8$.

In the proof we gave for the derivative of $[f(x)]^n$, we assumed that n is a positive integer, but it does not have to. It will soon be proved that

- *If r is any real number then:*

  $$\text{The derivative of } \; [f(x)]^r \; \textbf{is } r\,[f(x)]^{r-1}\cdot f'(x).$$

  This is called the generalized power rule for derivatives.

Example 2 *The derivative of* $u(x)=(5x-13)^{-3}$ *is*

$$u'(x)=-3\,(5x-13)^{-4}\,(5)=-15\,(5x-13)^{-4}.$$

In this case, n is a negative integer.

Example 3 *To determine the derivative of* $g(x)=\dfrac{4}{2x+5}$, *first write it as an exponent* $g(x)=4\,(2x+5)^{-1}$. *Then, by the general power rule,*

$$g'(x)=4\,(-1)\,(2x+5)^{-2}\,(2)=-8\,(2x+5)^{-2}.$$

Example 4 *To determine the derivative of* $v(x)=\sqrt{x^2+1}$, *first write it as* $v(x)=\left(x^2+1\right)^{1/2}$. *Then*

$$v'(x)=\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x)=x\left(x^2+1\right)^{-1/2}=\frac{x}{\sqrt{x^2+1}}$$

In this case n *is a positive rational number.*

Example 5 *Let* $w(x)=\dfrac{1}{\sqrt{2x^3+5}}=\left(2x^3+5\right)^{-1/2}$. *Its derivative is*

$$w'(x)=-\frac{1}{2}\left(2x^3+5\right)^{-3/2}\cdot\left(6x^2\right)=-3x^2\left(2x+3\right)^{-3/2}$$

In this case n *is a negative rational number.*

Exercise 6

1. *Copy and complete the following table:*

Function $[f(x)]^n$	*In this case*		*Its derivative is*
a. $\left(x^3+1\right)^4$	$f(x)=x^3+1,\ n=4$	$f'(x)=3x^2$	$4\left(x^3+1\right)^3\left(3x^2\right)=12x^2\left(x^3+1\right)^3$
b. $\left(x^2+3x-4\right)^8$			
c. $\sqrt{x^2+2x+8}$			
d. $\left(\sqrt{x}+5\right)^{3/5}$			
e. $\sqrt{3+\sqrt{x}}$			
f. $\dfrac{1}{\left(x^4+2x^2+5\right)^3}$			
g. $\dfrac{1}{\sqrt{x^2+3x+9}}$			
h. $\sqrt{\dfrac{4x}{3}-\dfrac{2}{x}}$			
i. $\left(\sqrt{2x+1}+x^3\right)^{-2}$			

2. *Determine the derivative of each function and simplify as much as possible.*

(a) $f(x)=(5x+2)^3$

(b) $g(x)=4\left(\frac{1}{2}x+4\right)^2$

(c) $h(x)=\frac{3}{4}\left(\sqrt{x^2+x}\right)$

(d) $u(t)=\dfrac{\sqrt{2t+1}}{3}$

(e) $v(s)=3\left(2s-3\right)^4$

(f) $w(x)=\frac{2}{5}\left(\frac{3}{5}x+1\right)^2$

(g) $f(y)=\sqrt{y}-\sqrt{2y+1}$

(h) $g(x)=\dfrac{2}{3x^2}-\left(3x+\frac{1}{2}\right)^3$

(i) $h(t)=\frac{1}{2}t^2-\dfrac{1}{\sqrt{2t^2+3}}$

(j) $u(x)=\left(2x^2+1\right)^2+3x^3$

(k) $w(y)=(3y-2)^2-(2y+3)^3$

(l) $f(x)=\sqrt{x^3+2}-\sqrt{3x^2-1}$

3. *A student gave the derivative of* $f(x) = \left(\frac{3x^2}{7} - 2x\right)^{\frac{2}{3}}$ *as* $f'(x) = \left(\frac{3x^2}{7} - 2x\right)^{-\frac{1}{3}} \left(\frac{6x}{7} - 2\right)^{-\frac{1}{3}}$. *What is the error?*

4. *Fill in the missing details in the following proof that if n is a positive integer then the derivative of* $f(x) = x^n$ *is* $f'(x) = nx^{n-1}$:

   We have to determine $\lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$. *Using the formula for* $X^n - Y^n$ *gives*

$$\begin{aligned} \frac{(x+h)^n - x^n}{h} &= \frac{[x+h-x]\left[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1}\right]}{h} \\ &= \left[(x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1}\right]. \end{aligned}$$

   Complete the exercise.

5. *You have to prove that if n is a positive integer then the derivative of* $g(x) = \frac{1}{x^n}$ *is* $-nx^{-n-1} = -\frac{n}{x^{n+1}}$. *To this end write g as* $g(x) = \left(\frac{1}{x}\right)^n$. *Since the derivative of* $f(x) = \frac{1}{x}$ *was shown to be* $-\frac{1}{x^2}$, *the general power rule implies that the derivative of g is* $g'(x) = n\left(\frac{1}{x}\right)^{n-1}\left(-\frac{1}{x^2}\right)$. *Show that this simplifies to* $-nx^{-n-1} = -\frac{n}{x^{n+1}}$.

6. *Let n be a positive integer and* $f(x) = x^{\frac{1}{n}}$. *Prove that* $f'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$. *More precisely, you have to show that* $\lim_{h\to 0} \frac{(x+h)^{\frac{1}{n}} - x^{\frac{1}{n}}}{h} = \frac{1}{n}x^{\frac{1}{n}-1}$. *To this end, denote* $(x+h)^{\frac{1}{n}}$ *by V and* $x^{\frac{1}{n}}$ *by U. Then* $V^n = (x+h)$ *and* $U^n = x$. *Note that*

$$V^n - U^n = (V-U)\left(V^{n-1} + V^{n-2}U + \cdots + VU^{n-2} + U^{n-1}\right)$$

   Since $V^n - U^n = h$, *it follows from the above identity that*

$$\begin{aligned} V - U &= \frac{V^n - U^n}{(V^{n-1} + V^{n-2}U + \cdots + VU^{n-2} + U^{n-1})} \\ &= \frac{h}{(V^{n-1} + V^{n-2}U + \cdots + VU^{n-2} + U^{n-1})}. \end{aligned}$$

   Use this to show that

$$\frac{(x+h)^{\frac{1}{n}} - x^{\frac{1}{n}}}{h} = \frac{1}{V^{n-1} + V^{n-2}U + \cdots + VU^{n-2} + U^{n-1}}$$

   then deduce that

$$\lim_{h\to 0} \frac{(x+h)^{\frac{1}{n}} - x^{\frac{1}{n}}}{h} = \frac{1}{nx^{\frac{n-1}{n}}} = \frac{1}{n}x^{\frac{1}{n}-1}$$

7. *Let m and n be integers and* $f(x) = x^{\frac{m}{n}}$. *Prove that* $f'(x) = \frac{m}{n}x^{\frac{m}{n}-1}$. *Hint: write the given function in the form* $f(x) = \left(x^{\frac{1}{n}}\right)^m$. *Then by the power rule and what has been proved above, the derivative of f is*

$$f'(x) = m\left(x^{\frac{1}{n}}\right)^{m-1}\left(\frac{1}{n}x^{\frac{1}{n}-1}\right)$$

   Show that this simplifies to $\frac{m}{n}x^{\frac{m}{n}-1}$.

The Product Rule

The product rule for derivatives enables us to determine the derivative of a product $f(x)g(x)$ if we know the derivatives of $f(x)$ and $g(x)$. Contrary to what you would hope for, the derivative of $f(x)g(x)$ is NOT the product $f'(x)g'(x)$ of the individual derivatives. Here is a counterexample:

Example 7 *Let $f(x) = x$ and $g(x) = 3x + 5$. Their product is $f(x) = f(x)g(x) = 3x^2 + 5x$ with derivative $f'(x) = 6x + 5$. On the other hand, $f'(x) = 1$, $g'(x) = 3$, and the product of their derivatives is $f'(x)g'(x) = (1)(3) = 3$ which bears no resemblances to the derivative of $f(x)g(x)$.*

The correct formula for the derivative of a product is:

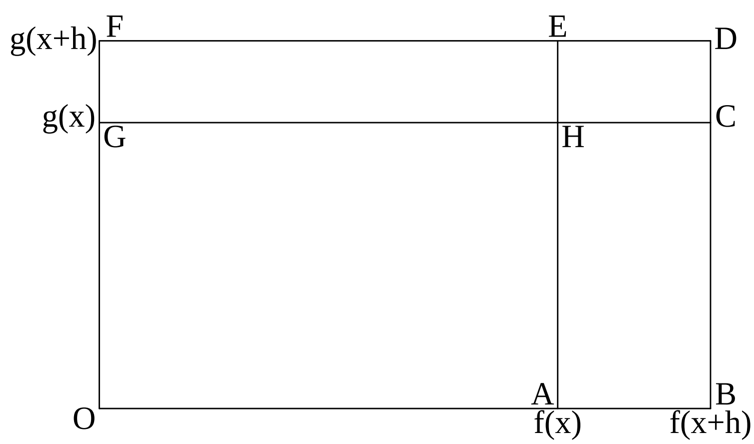
The derivative of $f(x)g(x)$ **is** $f'(x)g(x) + g'(x)f(x)$.

Applying it to the product of $f(x) = x$ and $g(x) = 3x+5$ gives the derivative of $f(x)g(x)$ to be $(1)(3x+5) + (3)(x) = 3x + 5 + 3x = 6x + 5$ as it should be.

To prove it, we have to show that

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = f'(x)g(x) + g'(x)f(x).$$

To this end consider the rectangles in the figure below. The lengths of OA, OB, OG and OF are $f(x)$, $f(x+h)$, $g(x)$, and $g(x+h)$ respectively. It follows that $f(x+h)g(x+h)$ is the area of rectangle OBDF and $f(x)g(x)$ is the area of rectangle OAHG.



Therefore

$$f(x+h)g(x+h) - f(x)g(x) = \text{Area of ABCH} + \text{Area of EFGH} + \text{Area of CDEH}$$

This implies that

$$\begin{aligned}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \frac{[f(x+h) - f(x)]\, g(x)}{h} + \frac{[g(x+h) - g(x)]\, f(x)}{h} \\ &+ \frac{[f(x+h) - f(x)]\,[g(x+h) - g(x)]}{h}\end{aligned}$$

the right hand side may be written as

$$\frac{f(x+h)-f(x)}{h}g(x)+\frac{g(x+h)-g(x)}{h}f(x)+\frac{f(x+h)-f(x)}{h}\left[g(x+h)-g(x)\right]$$

Since $\dfrac{f(x+h)-f(x)}{h}$, $\dfrac{g(x+h)-g(x)}{h}$, and $[g(x+h)-g(x)]$ have limits $f'(x)$, $g'(x)$ and 0 as respectively as h approaches 0, it follows that

$$\frac{f(x+h)-f(x)}{h}g(x)+\frac{g(x+h)-g(x)}{h}f(x)+\frac{f(x+h)-f(x)}{h}\left[g(x+h)-g(x)\right]$$

has limit $f'(x)g(x)+g'(x)f(x)+f'(x)\cdot 0 = f'(x)g(x)+g'(x)f(x)$ as h approaches 0. In other words,

$$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}=f'(x)g(x)+g'(x)f(x).$$

Exercise 8

1. *Determine the derivative of each function:*

   (a) $f(x)=2x(3x-1)^{-1}$

   (b) $g(x)=x\sqrt{x^2+2}$

   (c) $u(x)=\dfrac{x+1}{x+2}$ *(Hint: write as* $(x+1)(x+2)^{-1}$ *then use the product rule.)*

   (d) $w(x)=\dfrac{x}{3x-2}$

   (e) $v(x)=(2x+3)^{\frac{3}{2}}(5x+1)^{\frac{1}{2}}$

   (f) $q(x)=\dfrac{4x}{\sqrt{2x+1}}$

2. *Show that the derivative of* $f(x)=(3x+1)^5(5x-2)^3$ *is* $f'(x)=15(8x-1)(3x+1)^4(5x-2)^2$

3. *The product rule may be extended to a product* $f(x)g(x)w(x)$ *of three functions by writing it as* $f(x)[g(x)w(x)]$*, which we may regard as a product of two functions* $f(x)$ *and* $[g(x)w(x)]$*. Use the product rule twice to show that the derivative of* $f(x)g(x)w(x)$ *is*

$$f'(x)g(x)w(x)+g'(x)f(x)w(x)+w'(x)f(x)g(x)$$

4. *Use the above result to find the derivative of* $u(x)=3x\left(\sqrt{2x+5}\right)(3x-1)^{-1}$.

5. *You are given a quotient* $\dfrac{f(x)}{g(x)}$.

   (a) *Write it as the product* $f(x)[g(x)]^{-1}$ *and apply the product rule to show that:*

$$\text{The derivative of } \frac{f(x)}{g(x)} \text{ is } \frac{f'(x)g(x)-g'(x)f(x)}{[g(x)]^2}$$

   (b) *Use the result of part (a) to find the derivative of* $u(x)=\dfrac{x^2+1}{2x+5}$.