Review of Radicals

The exponents in many engineering, science, business, and technology problems will occur as real numbers, which need not be integers. When they occur as rational numbers we call them radicals. You have already encountered some, like square roots, (exponent $\frac{1}{2}$), cube roots, (exponent $\frac{1}{3}$), in general as nth roots, (exponent $\frac{1}{n}$). The following are a few examples of problemms that will generate radicals:

- Determining the distance between two points.
- Computing lengths of plane curves.
- The accumulated value of an amount of money earning compound interest.
- Finding geometric mean of numbers.
- Determining the mass of certain objects.

The formal definition of radicals is the following:

Let n be a positive integer bigger than 1, and a be a real number.

- If $a = 0$, then $\sqrt[n]{a} = 0$.
- If a is positive, then the n**th root of** a, denoted by $a^{\frac{1}{n}}$ or $\sqrt[n]{a}$, is the positive number that gives a when it is raised to power n. More briefly, it is the positive number b such that $b^n = a$.

  Example 1

  1. *The* 4*th root of* 81 *is the positive number* b *such that* $b^4 = 81$. *You probably guessed that* $b = 3$ *since* $3^4 = 81$.
  2. $10^{\frac{1}{5}}$ *or* $\sqrt[5]{10}$, *is the positive number* b *such that* $b^5 = 10$. *This time* b *is not an integer or a simple rational number. We have to use a calculator to determine its approximate value. It gives the value*1.58489 *rounded to 5 decimal places.*
  3. $\sqrt[3]{\frac{1}{8}} = \frac{1}{2}$, *because* $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

- If a is *negative* and n is odd, then $a^{\frac{1}{n}}$, also written as $\sqrt[n]{a}$, is the *negative* number that gives b when it is raised to power n. More precisely, it is the negative number b such that $b^n = a$.

  Example 2

  1. $\sqrt[5]{-32}$ *is the negative number* b *such that* $b^5 = -32$. *This is easy to guess and it is* -2.
  2. $\sqrt[3]{-64} = -4$
  3. $\sqrt[7]{-60}$ *is the negative number* b *such that* $b^7 = -60$. *We need a calculator to give its approximate value. It gives the value* -1.79482, *rounded to 5 decimal place*

- If a is negative and n is even, then $\sqrt[n]{a}$ is not a real number. It is a complex number. For example, $\sqrt[4]{-16}$ is a complex number.
- If $n = 2$ then $\sqrt[2]{a}$ is written simply as $\sqrt{a}$ and is called the *principal square root* of a, or simply the *square root* of a.

Raising a Radical to a Power / Taking a Radical of a Power

Consider a radical $a^{\frac{1}{n}}$. When we square it, (i.e. raise it to power 2), the result is

$$\left(a^{\frac{1}{n}}\right)^2 = \left(a^{\frac{1}{n}}\right)\left(a^{\frac{1}{n}}\right) \tag{1}$$

The rules of indices apply to radicals. When we apply them to (1) we get

$$\left(a^{\frac{1}{n}}\right)^2 = \left(a^{\frac{1}{n}}\right)\left(a^{\frac{1}{n}}\right) = a^{\frac{1}{n}+\frac{1}{n}} = a^{\frac{2}{n}}$$

Similarly,

$$\left(a^{\frac{1}{n}}\right)^3 = \left(a^{\frac{1}{n}}\right)\left(a^{\frac{1}{n}}\right)\left(a^{\frac{1}{n}}\right) = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}} = a^{\frac{3}{n}}$$

In general, if m is a positive integer then $\left(a^{\frac{1}{n}}\right)^m$ is the product of m terms each one equal to $\left(a^{\frac{1}{n}}\right)$. The result of such a multiplication is

$$\left(a^{\frac{1}{n}}\right)^m = \left(a^{\frac{1}{n}}\right)\left(a^{\frac{1}{n}}\right)\cdots\left(a^{\frac{1}{n}}\right) = a^{\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}} = a^{\frac{m}{n}}$$

This is one interpretation of $a^{\frac{m}{n}}$. In other words

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m \tag{2}$$

Another one is obtained by noting that $\sqrt[n]{a^m}$ is the number that gives a^m when it is raised to power n. Since

$$\underbrace{\frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n}}_{\text{a total of n terms}} = m,$$

it follows that $\left(a^{\frac{m}{n}}\right)^n = a^{\frac{m}{n}+\frac{m}{n}+\cdots+\frac{m}{n}} = a^m$. In other words, when we raise $a^{\frac{m}{n}}$ to power m the result is a^m. Therefore $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Example 3

1. $4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)\left(4^{\frac{1}{2}}\right)\left(4^{\frac{1}{2}}\right) = (2)(2)(2) = 8$. *Alternatively* $4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$
2. $125^{\frac{2}{3}} = \left(125^{\frac{1}{3}}\right)\left(125^{\frac{1}{3}}\right) = (5)(5) = 25$. *Alternatively* $125^{\frac{2}{3}} = \sqrt[3]{125^2} = \sqrt[3]{15625} = 25$.
3. $81^{\frac{3}{4}} = \left(81^{\frac{1}{4}}\right)\left(81^{\frac{1}{4}}\right)\left(81^{\frac{1}{4}}\right) = (3)(3)(3) = 27$. *Alternatively* $81^{\frac{3}{4}} = \sqrt[4]{81^3} = \sqrt[4]{531441} = 27$

If we take $m = n$ in (2), we get $a^{\frac{n}{n}} = \left(a^{\frac{1}{n}}\right)^n$. Since $a^{\frac{n}{n}} = a^1 = a$, we conclude that

$$\left(a^{\frac{1}{n}}\right)^n = a$$

This is in perfect agreement with the definition of $a^{\frac{1}{n}}$ as the number one has to raise to power n to get a. The result may also be written as $\left(\sqrt[n]{a}\right)^n = a$

Laws of Radicals:

1. If $\sqrt[n]{a}$ is defined then $\left(\sqrt[n]{a}\right)^n = a$.

   Example: $\left(\sqrt{3}\right)^2 = 3$ and $\left(\sqrt[3]{-27}\right)^3 = -27$

2. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are defined then $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$

Examples: $\sqrt{4\times 25}=\sqrt{4}\sqrt{25}=2\times 5=10$

$$\sqrt[3]{-32}=\sqrt[3]{(-8)(4)}=\sqrt[3]{(-8)}\sqrt[3]{4}=-2\left(\sqrt[3]{4}\right)$$

$$\sqrt{600}=\sqrt{4\times 6\times 25}=\sqrt{4}\sqrt{6}\sqrt{25}=10\sqrt{6}$$

3. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are defined and b is not 0 then $\sqrt[n]{\dfrac{a}{b}}=\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

   Example: $\sqrt[3]{\dfrac{8}{27}}=\dfrac{\sqrt[3]{8}}{\sqrt[3]{27}}=\dfrac{2}{3}$

4. If m and n are positive integers and a is positive then $\left(a^{\frac{1}{n}}\right)^{\frac{1}{m}}=a^{\frac{1}{mn}}$ This may also be written as $\sqrt[m]{\sqrt[n]{a}}=\sqrt[mn]{a}$.

   Example: $\sqrt{\sqrt[3]{64}}=\sqrt[2]{\sqrt[3]{64}}=\sqrt[(2)(3)]{64}=\sqrt[6]{2^6}=2$.

   CAUTION: Whereas the law $\sqrt[n]{ab}=\sqrt[n]{a}\sqrt[n]{b}$ is true for products, IT IS NOT TRUE for sums or differences as indicated below:

$$\sqrt{16+9}\neq\sqrt{16}+\sqrt{9}$$

   since the left hand side is $\sqrt{25}$ which is 5, while the right hand side is $4+3$ which is 7.

$$\sqrt{169-25}\neq\sqrt{169}-\sqrt{25}$$

   since the left hand side is $\sqrt{144}$ which equals 12 while the right hand side is $13-5$ which equals 8.

5. If a and b are positive numbers then $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=a-b$. This may be used to rationalize numerators of denominators.

Example 4 *The following illustrate how we may simplify radicals. For simplicity, assume that all the variables are positive.*

1. $\sqrt[3]{40}=\sqrt[3]{8\times 5}=\sqrt[3]{8}\sqrt[3]{5}=2\sqrt[3]{5}$
2. $\sqrt[5]{y^9}=\sqrt[5]{y^5y^4}=\sqrt[5]{y^5}\sqrt[5]{y^4}=y\left(\sqrt[5]{y^4}\right)$
3. $\sqrt{x^4y^3}=\sqrt{x^4}\sqrt{y^3}=x^2\sqrt{y^2y}=x^2y\left(\sqrt{y}\right)$
4. $\sqrt{y^6}=\sqrt{\left(y^3\right)^2}=y^3$
5. $\sqrt[3]{135}=\sqrt[3]{27\times 5}=\sqrt[3]{27}\sqrt[3]{5}=3\left(\sqrt[3]{5}\right)$
6. $4\sqrt{24}-\sqrt{54}=4\sqrt{4\times 6}-\sqrt{9\times 6}=4\times 2\sqrt{6}-3\sqrt{6}=8\sqrt{6}-3\sqrt{6}=5\sqrt{6}$
7. $\sqrt{\dfrac{32x^4y^3z^6}{27w^2}}=\dfrac{\sqrt{32}\sqrt{x^4}\sqrt{y^3}\sqrt{z^6}}{\sqrt{27}\sqrt{w^2}}=\dfrac{\left(4\sqrt{2}\right)\left(x^2\right)\left(y\sqrt{y}\right)\left(z^3\right)}{3\sqrt{3}\,(w)}=\dfrac{4x^2yz^3}{3w}\sqrt{\dfrac{2y}{3}}$
8. $\sqrt{5x^2y^5}\sqrt{8x^7y}=\sqrt{5x^2y^4\,(4)\,(2)\,x^6xy}=2x^4y^2\sqrt{10xy}$
9. $27^{\frac{2}{3}}=\left(27^{\frac{1}{3}}\right)^2=(3)^2=9$

10. $\left(\frac{32}{243}\right)^{\frac{2}{5}} = \left(\left(\frac{32}{243}\right)^{\frac{1}{5}}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}.$

Example 5 *To rationalize the denominator of* $\frac{3}{\sqrt{7}+\sqrt{2}}$

To make use of Law 5, multiply the numerator and denominator of the given expression by $\left(\sqrt{7}-\sqrt{2}\right)$. *The result is*

$$\frac{3}{\sqrt{7}+\sqrt{2}} = \frac{3}{\sqrt{7}+\sqrt{2}} \times \frac{\sqrt{7}-\sqrt{2}}{\sqrt{7}-\sqrt{2}} = \frac{3(\sqrt{7}-\sqrt{2})}{7-2} = \frac{3(\sqrt{7}-\sqrt{2})}{5}.$$

Thus $\frac{3}{\sqrt{7}+\sqrt{2}} = \frac{3(\sqrt{7}-\sqrt{2})}{5}$ *and the right hand side has a rational denominator.*

Example 6 *To rationalize the numerator of* $\frac{\sqrt{x+h}-\sqrt{x}}{h}$.

Multiply the numerator and denominator by $\sqrt{x+h}+\sqrt{x}$. *The result is*

$$\begin{aligned}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

The numerator is now rational.

Example 7 *To rationalize the denominator of* $\frac{1}{\sqrt{7}}$:

Multiply the numerator and denominator by $\sqrt{7}$ *(to get* $\sqrt{7}\sqrt{7}$ *in the denominator). The result is*

$$\frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{\sqrt{7}\sqrt{7}} = \frac{\sqrt{7}}{\sqrt{7^2}} = \frac{\sqrt{7}}{7}.$$

The denominator of $\frac{\sqrt{7}}{7}$ *is rational.*

Example 8 *To rationalize the denominator of* $\frac{1}{\sqrt[5]{y}}$:

Multiply the numerator and denominator by $\sqrt[5]{y^4}$ *(to get* $\left(\sqrt[5]{y}\right)\left(\sqrt[5]{y^4}\right) = \sqrt[5]{y^5}$ *in the denominator). The result is*

$$\frac{1}{\sqrt[5]{y}} = \frac{\sqrt[5]{y^4}}{\left(\sqrt[5]{y}\right)\left(\sqrt[5]{y^4}\right)} = \frac{\sqrt[5]{y^4}}{\sqrt[5]{y^5}} = \frac{\sqrt[5]{y^4}}{y}.$$

Exercise 9

1. *Simplify:*

   (a) $\left(5x^{\frac{2}{3}}\right)\left(3x^{\frac{1}{3}}\right)$ *(b)* $\left(-4x^{\frac{9}{5}}\right)\left(7x^{\frac{6}{5}}\right)$ *(c)* $\left(-6x^{\frac{7}{3}}\right)\left(2x^{\frac{4}{3}}\right)$

   (d) $3\left(x^4\right)^{-\frac{7}{4}}$ *(e)* $\left(-5x^{\frac{1}{3}}\right)\left(7x^{\frac{4}{3}}\right)$ *(f)* $\left(\frac{-32x^5}{y^{-10}}\right)^{\frac{4}{5}}$

   (g) $x^{2/3}x^{-1/3}x^{-4/3}$ *(h)* $64^{2/3} - 25^{3/2}$ *(i)* $\left(\frac{4}{9}\right)^{1/2} + \left(\frac{125}{64}\right)^{-2/3}$

2. *Rewrite the following expressions by changing to rational exponents:*

(a) $\sqrt[5]{x^2}$ (b) $\sqrt[3]{x^5}$ (c) $\sqrt[4]{(x+y)^3}$

(d) $\sqrt{x+\sqrt[3]{y^2}}$ (e) $\sqrt[3]{(x-y)^4}$ (f) $\sqrt[6]{x^3+y^3}$

3. *Rewrite the following expressions by changing to radicals:*

(a) $15x^{5/2}$ (b) $(7y)^{7/3}$ (c) $12-x^{5/3}$

(d) $\left(12-x^3\right)^{5/3}$ (e) $27x^{1/3}$ (f) $(27x)^{1/3}$

4. *Simplify the following expressions and rationalize the denominator if it is not rational:*

(a) $\sqrt{81}$ (b) $\sqrt[3]{-64}$ (c) $\dfrac{1}{\sqrt[5]{3}}$

(d) $\dfrac{3}{\sqrt{5}}$ (e) $\sqrt{25x^{-6}y^2}$ (f) $\sqrt[3]{27x^4y^9}$

(g) $\sqrt[5]{\dfrac{x^8y^7}{32x^2}}$ (h) $\sqrt[4]{(5x^6y^{-3})}$ (i) $\sqrt{3x^2y^5}\sqrt{12x^4y^2}$

(j) $\sqrt[4]{4\sqrt{8}+8}$ (k) $\sqrt[3]{27}\sqrt[3]{9}\sqrt[3]{3}$ (l) $\sqrt[3]{\dfrac{(243000)(0.008)}{1.25}}$

(m) $\sqrt[3]{108}-3\sqrt[3]{32}$ (n) $\sqrt{\sqrt[3]{64}}+\sqrt[3]{\sqrt{729}}$ (o) $\sqrt{44}+\sqrt{99}+\sqrt{176}$

(p) $\sqrt{144+25}$ (q) $2\sqrt[3]{243}-6\sqrt[3]{-72}$ (r) $\dfrac{\sqrt[3]{81}}{\sqrt[3]{3}}+\dfrac{\sqrt[7]{128}}{\sqrt[2]{4}}-\dfrac{\sqrt{36}}{\sqrt[3]{27}}$

5. *Use a calculator to evaluate each exponent to 4 decimal places.*

(a) $15.9^{1.12}$ (b) $7.98^{-2.7}$ (c) $160.2^{0.45}$
(d) $9.84^{5/3}$ (e) $27^{3/4}$ (f) 1.028^{63}

6. *The distance D (in miles) that can be seen, on a clear day, from the top of a high rise building of height H (in feet) is approximately computed by the formula*

$$D = 1.2\sqrt{H}$$

Find the distance that can be seen on a clear day from the top of the Empire State building, which is approximately 1390 feet.

7. *The body surface area (S) of a person weighing W pounds and H inches tall can be approximates by the formula*

$$S = (0.1091)\,W^{0.425}H^{0.725}.$$

Find the surface area of a person 5 feet tall, weighing 180 pounds.

8. *The body-mass index B for a person weighing W pounds and who is H inches tall can be determined by the formula* $B = \dfrac{703W}{H^2}$. *Determine the body-mass index of a person who weighs 230 pounds and is 6 ft. 5 in. tall.*