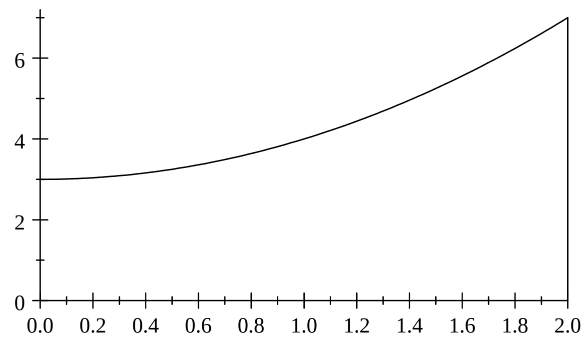
Making a Substitution in an Integral

There are integrals which may not yield to integration by inspection. An example is

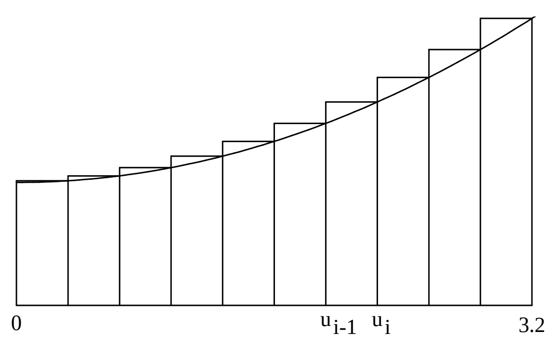
$$\int_3^9 \frac{x^3}{(x+1)^2}\,dx.$$

In such a case, we change to a new variable, a process called "substitution", to convert the unfamiliar integral into a familiar one.

If you are willing to over-simplify, then making substitution may be likened to "changing" from one unit of measure into another, (e.g. from miles into kilometers), in a problem of computing areas. For a specific case, go back to the problem we solved, of determining the area of the piece of land between a straight road and a river shaped like the graph of the parabola $f(x) = x^2 + 3,\ \ 0 \le x \le 2$, where x is measured in miles.



Say it is given to an individual who is used to measuring distances in kilometers, and he insists on measuring distances along the road in kilometers. Then, according to his calculations, the land is between the 0 and the $\frac{16}{5} = 3.2$ kilometer marks, (because 1 mile is equal to $\frac{8}{5}$ kilometers). He proceeds to divide it into small strips using subintervals $[0, u_1]$, $[u_1, u_2]$, $\ldots$, $[u_{n-2}, u_{n-1}]$, $\left[u_{n-1}, \frac{16}{5}\right]$ of $\left[0, \frac{16}{5}\right]$, then approximates each strip with a rectangle as shown below.



The expression $\left(x^2+3\right)$ gives the correct height of a rectangle only if x is in miles. Therefore, to calculate the height of a strip on a typical interval $[u_{i-1}, u_i]$, we first have to convert u_i kilometers into miles. Since 1 kilometer is $\frac{5}{8}$ miles, the height is

$$\left[\left(\frac{5u_i}{8}\right)^2 + 3\right] \quad \text{miles.}$$

The width and height should be measured in the same units, therefore we must convert the $(u_i - u_{i-1})$ kilometers for the width into $\left(\frac{5}{8}\right)(u_i - u_{i-1})$ miles. Then the area of the typical rectangle is

$$\left[\left(\tfrac{5u_i}{8}\right)^2 + 3\right]\left(\tfrac{5}{8}\right)(u_i - u_{i-1}) \ \text{ square miles.}$$

It follows that the area of the piece of land is the limit of the Riemann sums

$$\sum_{i=1}^{n}\left[\left(\tfrac{5}{8}u_i\right)^2+3\right]\tfrac{5}{8}\left(u_i-u_{i-1}\right)$$

That limit is

$$\int_0^{16/5}\left[\left(\tfrac{5}{8}u\right)^2+3\right]\tfrac{5}{8}du$$

We say that we have made a substitution $x=\frac{5}{8}u$ in the integral

$$\int_0^2\left(x^2+3\right)dx.$$

This boiled down to replacing x by $\frac{5}{8}u$ and a little bit more. We had to replace interval $[0,2]$ for the variable x by $[0,3.2]$ for the new variable u. We also had to replace dx by $\frac{5}{8}du$, a step corresponding to giving the correct "measures" for the widths of the approximating rectangles. We will call $\frac{5}{8}$ the *scaling factor* for this substitution. Note that if we define the function $g(u)=\frac{5}{8}u$, then replacing x^2+3 by $\left(\frac{5}{8}u\right)^2+3$ is simply forming the composition $f\circ g(u)=f(g(u))$. Furthermore, the scaling factor $\frac{5}{8}$ is the derivative of g and $\left[0,\frac{16}{5}\right]$ is the interval that g maps onto $[0,2]$, (i.e. $g(0)=0$ and $g(\frac{16}{5})=2$). Therefore we have verified that

$$\int_0^2 f(x)dx=\int_0^{16/5}f(g(u))g'(u)du$$

To generalize, let f be a given function and $[a,b]$ be an interval. To make a substitution in the integral $\int_a^b f(x)dx$, we do the following:

1. *Replace the independent variable x with a new variable u.* (We likened this to changing from one set of units to another.) *This amounts to composing f with some suitable function g to get a new function $f(g(u))$.*
2. *Multiply $f(g(u))$ by a scaling factor which happens to be $g'(u)$.*
3. *Determine the interval $[c,d]$ that g maps onto $[a,b]$. Then*

$$\int_a^b f(x)dx=\int_c^d f(g(u))g'(u)du \tag{1}$$

We will soon defend (1). To apply it successfully, one should choose $g(u)$ such that $\int_c^d f(g(u))g'(u)du$ can be determined by inspection. A bad choice for $g(u)$ may give another unfamiliar integral $\int_c^d f(g(u))g'(u)du$.

Example 1 *Say we wish to integrate* $\int_0^4 \frac{x}{\sqrt{2x+1}}dx$ *by substitution: The integrand is*

$$f(x)=\frac{x}{\sqrt{2x+1}}.$$

We have to compose it with a suitable function g chosen so that $f(g(u))g'(u)$ can be integrated by inspection. To get it, note that if the denominator of the integrand were $\sqrt{u}$ and the numerator was a polynomial in u then we would divide each term in the numerator by $\sqrt{u}$ and proceed to integrate by inspection. Therefore it is reasonable to look for a function g with the property that when we form the composition $f(g(u))$, the expression $\sqrt{2x+1}$ becomes $\sqrt{u}$. The choice $g(u)=\left(\frac{u-1}{2}\right)$ does precisely that. (We get it by setting $u=2x+1$, so that $\sqrt{2x+1}$ becomes $\sqrt{u}$, then solve for x in terms of u.) The scaling factor is $g'(u)=\frac{1}{2}$, therefore

$$f(g(u))g'(u)=\frac{1}{\sqrt{u}}\left(\frac{u-1}{2}\right)\cdot\tfrac{1}{2}=\tfrac{1}{4}\left(u^{1/2}-u^{-1/2}\right)$$

which we can integrate by inspection. We need the interval that g maps onto $[0,2]$. Since g is increasing, we have to find numbers c and d such that $g(c)=0$ and $g(d)=4$. They are $c=1$ and $d=9$, (obtained by solving $\frac{u-1}{2}=0$ and $\frac{u-1}{2}=4$). Therefore,

$$\int_0^4 \frac{x}{\sqrt{2x+1}}dx = \int_1^9 \tfrac{1}{4}\left(u^{1/2}-u^{-1/2}\right)du = \tfrac{1}{4}\left[\tfrac{2}{3}u^{3/2}-2u^{1/2}\right]_1^9 = \tfrac{10}{3}.$$

The following are short-cuts that lead to the same result. They are defended ahead.

- Instead of defining $g(u)=\left(\dfrac{u-1}{2}\right)$ then form the composition $f(g(u))$, simply let $u=2x+1$ and substitute it into the expression $\dfrac{x}{\sqrt{2x+1}}$ to get $\dfrac{x}{\sqrt{u}}$. (In fact if we define $g(u)=\left(\dfrac{u-1}{2}\right)$ and form the composition $f(g(u))$, $\sqrt{2x+1}$ is replaced by $\sqrt{u}$.)
- Differentiate u with respect to x. The result is $\dfrac{du}{dx}=2$. Now regard $\dfrac{du}{dx}$ as fraction and solve to get $dx=\frac{1}{2}du$. The scaling factor is $\frac{1}{2}$ and we replace dx by $\frac{1}{2}du$. (We have changed to a new independent variable u hence the symbol du.) Indeed $g'(u)=\frac{1}{2}$ by direct verification.
- Replace the interval $[0,2]$ with $[u(0),u(2)]=[1,9]$.
- Then $\displaystyle\int_0^2 \frac{x}{\sqrt{2x+1}}dx = \int_1^9 \frac{x}{2\sqrt{u}}du = \frac{1}{2}\int_1^9 \frac{x}{\sqrt{u}}du$. Since $u=2x+1$, replace x with $\frac{1}{2}(u-1)$. Therefore

$$\tfrac{1}{2}\int_1^9 \frac{x}{\sqrt{u}}du = \tfrac{1}{2}\int_1^9 \frac{\frac{1}{2}(u-1)}{\sqrt{u}}du = \tfrac{1}{4}\int_1^9 \left(u^{1/2}-u^{-1/2}\right)du.$$

  The rest is similar to what we have done above.

In general, to make a substitution in a given integral $\displaystyle\int_a^b f(x)dx$ using the short-cuts we do the following:

1. Pick an expression in the formula for $f(x)$ and denote it by u. This is called making a substituting $u=$ (expression in x). It is a trial and error process. You will know that you have made an incorrect choice if you do not get a simpler integral after the substitution.
2. Determine $u'=\dfrac{du}{dx}$ then regard $\dfrac{du}{dx}$ as a fraction with numerator du and denominator dx and solve for dx to get $dx=\dfrac{1}{u'}du$. Now replace dx by $\dfrac{1}{u'}du$.
3. Use the relation $u=$ (expression in x) to get rid of the variable x in the resulting expression $f(x)\dfrac{1}{u'}du$. Denote what you get by

   $$(\textit{Equivalent expression in the variable } u)du.$$

4. Replace the lower limit a of integration by $u(a)$, (the value of u when $x=a$) and the upper limit by $u(b)$, (the value of u when $x=b$). Then

   $$\int_a^b f(x)dx = \int_{u(a)}^{u(b)} (\textit{Equivalent expression in the variable } u)du.$$

Example 2 *To integrate $\displaystyle\int_0^3 x^2\sqrt{5x+1}dx$ by substitution, using the short-cuts, we let $u=5x+1$. Then $\dfrac{du}{dx}=5$, $dx=\frac{1}{5}du$ and $x=\frac{1}{5}(u-1)$. Substitute these into $f(x)dx=x^2\sqrt{5x+1}dx$ to get*

$$f(x)\frac{1}{u'}du = \tfrac{1}{5}x^2\sqrt{u}du = \tfrac{1}{5}\left[\tfrac{1}{5}(u-1)\right]^2\sqrt{u}du = \tfrac{1}{125}\left(u^{5/2}-2u^{3/2}+u^{1/2}\right)du.$$

In this case "(Equivalent expression in the variable u)du" is

$$\tfrac{1}{125}\left(u^{5/2}-2u^{3/2}+u^{1/2}\right)du.$$

When $x=0$, $u=1$ *and when* $x=3$, $u=4$, *therefore*

$$\begin{aligned}\int_0^3 x^2\sqrt{5x+1}dx &= \tfrac{1}{5}\int_1^4 \left[\tfrac{1}{5}(u-1)\right]^2\sqrt{u}du=\tfrac{1}{125}\int_1^4\left(u^{5/2}-2u^{3/2}+u^{1/2}\right)du\\ &= \tfrac{1}{125}\left[\tfrac{2}{7}u^{7/2}-\tfrac{4}{5}u^{5/2}+\tfrac{2}{3}u^{3/2}\right]_1^4=0.129 \text{ (to 3 decimal places).}\end{aligned}$$

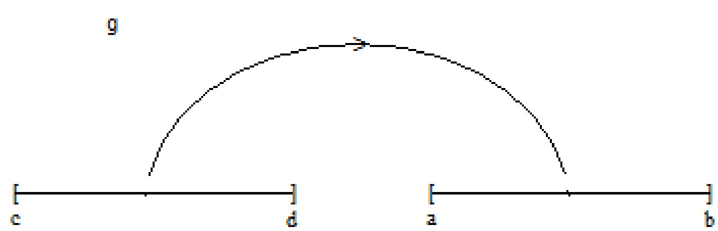
Justifying the short-cuts

Let $\int_a^b f(x)dx$ be a given integral. Say we have picked an expression $h(x)$ in the formula for $f(x)$ and have defined $u=h(x)$.

- If we choose $g=h^{-1}$, (the inverse of h) and form the composition $f(g(u))$, the expression $h(x)$ in the formula for f is replaced by u.
- Since $h(g(u))=u$, the chain rule implies that $h'(g(u))g'(u)=1$. Therefore

  $$g'(u)=\frac{1}{h'(x)}=\frac{1}{u'}$$

  It follows that multiplying $f(g(u))$ by $dx=\dfrac{1}{u'}du$ amounts to introducing the scaling factor $g'(u)$ for the integral.
- We have to find numbers c and d such that $a=g(c)$ and $b=g(d)$.



The diagram above tells the story. The function g maps the interval $[c,d]$ onto the interval $[a,b]$ in such a way that $g(c)=a$ and $g(d)=b$. But by definition, $g=h^{-1}$. Therefore the equation $g(c)=a$ may be written as $h^{-1}(c)=a$, which implies that $c=h(a)$. Similarly, $h^{-1}(d)=b$ implies that $d=h(b)$.

Therefore

$$\int_a^b f(x)dx=\int_{u(a)}^{u(b)} f(g(u))g'(u)du=\int_{u(a)}^{u(b)} f(g(u))\frac{1}{u'}du.$$

Example 3 *To determine* $\int_0^1 \dfrac{x^3}{(3x+2)^2}dx$, *let* $u=3x+2$. *Then* $\dfrac{du}{dx}=3$ *and so* $dx=\tfrac{1}{3}du$. *When* $x=0$,

$u=2$ and when $x=1$, $u=5$. Also, $x=\frac{1}{3}(u-2)$, therefore

$$\begin{aligned}\int_0^1 \frac{x^3}{(3x+2)^2}dx &= \int_2^5 \frac{(u-2)^3}{27u^2}\cdot\frac{1}{3}du = \frac{1}{81}\int_2^5\left(u-6+\frac{12}{u}-\frac{8}{u^2}\right)du\\ &= \tfrac{1}{81}\left[\tfrac{1}{2}u^2-6u+12\ln u+\frac{8}{u}\right]_2^5 = \tfrac{1}{81}(12\ln 2.5-9.9)\end{aligned}$$

Example 4 *To determine $\int_0^1 \frac{x^5}{\sqrt{x^2+1}}dx$, let $u=x^2+1$. Then $\frac{du}{dx}=2x$ and so $dx=\frac{1}{2x}du$. When $x=0$, $u=1$ and when $x=1$, $u=2$. Therefore*

$$\int_0^1\frac{x^5}{\sqrt{x^2+1}}dx=\int_1^2\frac{x^5}{\sqrt{u}}\cdot\frac{1}{2x}du=\tfrac{1}{2}\int_1^2\frac{x^4}{\sqrt{u}}du$$

Since $x^2=u-1$, we replace x^4 with $\left(x^2\right)^2=(u-1)^2=u^2-2u+1$. Therefore

$$\begin{aligned}\int_0^1\frac{x^5}{\sqrt{x^2+1}}dx &= \tfrac{1}{2}\int_1^2\frac{u^2-2u+1}{\sqrt{u}}du=\tfrac{1}{2}\int_1^2\left(u^{3/2}-2u^{1/2}+u^{-1/2}\right)du\\ &= \tfrac{1}{2}\left[\tfrac{2}{5}u^{5/2}-\tfrac{4}{3}u^{3/2}+2u^{1/2}\right]_1^2=\tfrac{7\sqrt{2}-8}{15}.\end{aligned}$$

Example 5 *To determine $\int\left(x^2\sqrt{4x+1}\right)dx$, let $u=4x+1$. Then $\frac{du}{dx}=4$, hence $dx=\frac{1}{4}du$. Also $x=\frac{1}{4}(u-1)$. Therefore*

$$\begin{aligned}\int\left(x^2\sqrt{4x+1}\right)dx &= \int\tfrac{1}{16}(u-1)^2\sqrt{u}\cdot\frac{du}{4}=\tfrac{1}{64}\int\left(u^{5/2}-2u^{3/2}+u^{1/2}\right)du\\ &= \tfrac{1}{64}\left(\tfrac{2}{7}u^{7/2}-\tfrac{4}{5}u^{5/2}+\tfrac{2}{3}u^{3/2}\right)+c\end{aligned}$$

Thus $\int\left(x^2\sqrt{4x+1}\right)dx=\frac{1}{64}\left(\frac{2}{7}(4x+1)^{7/2}-\frac{4}{5}(4x+1)^{5/2}+\frac{2}{3}(4x+1)^{3/2}\right)+c$

Example 6 *To find $\int\tan x\sqrt{\sec x}dx$, let $u=\sec x$. Then $\frac{du}{dx}=\sec x\tan x$ and $dx=\frac{du}{\sec x\tan x}$. The integral becomes*

$$\begin{aligned}\int\tan x\sqrt{\sec x}dx &= \int\sqrt{u}\cdot\frac{du}{\sec x}=\int\sqrt{u}\cdot\frac{du}{u}=\int\frac{1}{\sqrt{u}}du\\ &= 2u^{1/2}+c=2\sqrt{\sec x}+c.\end{aligned}$$

Example 7 *To determine $\int\sin^3x\cos^4xdx$, let $u=\cos x$. Then $\frac{du}{dx}=-\sin x$ and $dx=-\frac{1}{\sin x}du$. The integral becomes*

$$\int\sin^3x\cos^4xdx=-\int\left(\sin^3x\right)u^4\cdot\frac{1}{\sin x}du=-\int\left(\sin^2x\right)u^4du$$

To replace the term $\sin^2x$ by an expression involving u, we use the trigonometric identity $\sin^2x=1-\cos^2x=1-u^2$. Therefore

$$\begin{aligned}\int\sin^3x\cos^4xdx &= -\int\left(1-u^2\right)u^4xdx=\int\left(u^6-u^4\right)du=\frac{u^7}{7}-\frac{u^5}{5}+c\\ &= \tfrac{1}{7}\cos^7x-\tfrac{1}{5}\cos^5x+c.\end{aligned}$$

You should memorize the results of the next two examples.

Example 8 *To determine* $\int \cot x dx = \int \frac{\cos x}{\sin x} dx$*, let* $u = \sin x$*. Then* $\frac{du}{dx} = \cos x$ *and* $dx = \frac{1}{\cos x} du$*. Therefore*

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{\cos x}{u} \cdot \frac{1}{\cos x} du = \int \frac{1}{u} du = \ln |u| + c = \ln |\sin x| + c$$

I.e. $\int \cot x dx = \ln |\sin x| + c$

Example 9 *To determine* $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$*, let* $u = \cos x$*. Then* $\frac{du}{dx} = -\sin x$ *and* $dx = -\frac{1}{\sin x} du$*. Therefore*

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{\sin x}{u \sin x} du = -\int \frac{1}{u} du = -\ln |u| + c = \ln |\sec x| + c$$

I.e. $\int \tan x dx = \ln |\sec x| + c$

Example 10 *To determine* $\int \frac{1}{\sqrt{1-9x^2}} dx$*, write* $\sqrt{1-9x^2}$ *as* $\sqrt{1-(3x)^2}$*. Now the substitution* $u = 3x$ *converts the integrand into the familiar function* $\frac{1}{\sqrt{1-u^2}}$*. Therefore let* $u = 3x$*. Then* $\frac{du}{dx} = 3$ *and* $dx = \frac{du}{3}$*. The integral becomes*

$$\int \frac{1}{\sqrt{1-9x^2}} dx = \tfrac{1}{3} \int \frac{1}{\sqrt{1-u^2}} du = \tfrac{1}{3} \arcsin u + c = \tfrac{1}{3} \arcsin 3x + c$$

Example 11 *To determine* $\int \frac{1}{1+16x^2} dx$*, note that*

$$\frac{1}{1+16x^2} = \frac{1}{1+(4x)^2}$$

and the substitution $u = 4x$ *converts this into the familiar integrand* $\frac{1}{1+u^2}$*. Therefore let* $u = 4x$*. Then* $\frac{du}{dx} = 4$ *and* $dx = \frac{1}{4} du$*. The integral becomes*

$$\begin{aligned} \int \frac{1}{1+16x^2} dx &= \tfrac{1}{4} \int \frac{1}{1+u^2} du \\ &= \tfrac{1}{4} \arctan u + c = \tfrac{1}{4} \arctan 4x + c \end{aligned}$$

Example 12 *To determine* $\int \frac{1}{5+x^2} dx$*, we look for a substitution that changes* x^2 *into* $5u^2$ *so that we may factor out the 5. The choice* $x^2 = 5u^2$*, which is equivalent to*$\sqrt{5}u = x$*, does it. We find that* $\frac{du}{dx} = \frac{1}{\sqrt{5}}$*, hence* $dx = \sqrt{5} du$*, and the integral becomes*

$$\int \frac{1}{5+x^2} dx = \int \frac{\sqrt{5}}{5(1+u^2)} du = \tfrac{\sqrt{5}}{5} \arctan u + c = \tfrac{1}{\sqrt{5}} \arctan \left(\tfrac{x}{\sqrt{5}} \right) + c$$

Example 13 *To determine* $\int \frac{1}{a^2+b^2x^2} dx$ *where* a *and* b *are non-zero constants, we need a substitution that changes* b^2x^2 *into* a^2u^2*, (so we can factor out the* a^2*). The choice* $b^2x^2 = a^2u^2$*, which is equivalent to* $u = \frac{bx}{a}$*, does precisely that. We get* $\frac{du}{dx} = \frac{b}{a}$*, hence* $dx = \frac{a}{b} du$*, and the integral becomes*

$$\int \frac{1}{a^2+b^2x^2} dx = \tfrac{a}{b} \int \frac{1}{a^2(1+u^2)} du = \tfrac{1}{ab} \arctan u + c = \tfrac{1}{ab} \arctan \left(\tfrac{bx}{a} \right) + c.$$

Example 14 *To determine* $\int \frac{1}{\sqrt{a^2 - b^2x^2}}dx$ *where* a *and* b *are non-zero constants, we again use the substitution* $u = \frac{bx}{a}$ *to change* b^2x^2 *into* a^2u^2*. Then* $\frac{du}{dx} = \frac{b}{a}$ *and* $dx = \frac{a}{b}du$*. The integral becomes*

$$\int \frac{1}{\sqrt{a^2 - b^2x^2}}dx = \tfrac{a}{b}\int \frac{1}{a\sqrt{1-u^2}}dx = \tfrac{1}{b}\arcsin u + c = \tfrac{1}{b}\arcsin\left(\tfrac{bx}{a}\right) + c$$

Exercise 15

1. *Use the suggested substitution to determine the given integral*

   a) $\int_{-1}^{2} \frac{x}{\sqrt{x+2}}dx,\ u = x+2$ b) $\int_{0}^{8} x^3\sqrt{x+1}dx,\ u = x+1$ c) $\int_{5}^{10} \frac{x+1}{\sqrt{x-1}}dx,\ u = x-1$

   d) $\int_{0}^{\frac{\pi}{3}} (\tan x)(\sec x)^{\frac{3}{2}}\,dx,\ u = \sec x$ e) $\int \frac{1}{3+x^2}dx,\ \sqrt{3}u = x$ f) $\int x^2\sqrt{x+4}dx,\ u = x+4$

   g) $\int (x+1)\sqrt{x-2}dx,\ u = x-2$ h) $\int \frac{1}{\sqrt{1-3x^2}}dx,\ u = \sqrt{3}x$ i) $\int_{0}^{7} \frac{x^2}{\sqrt{x+1}}dx,\ u = x+1$

   j) $\int_{0}^{1} \frac{1-\sqrt{x}}{1+\sqrt{x}}dx,\ u = 1+\sqrt{x}$ k) $\int_{-2}^{13} \frac{x^2}{\sqrt{x+3}}dx,\ u = x+3$ l) $\int \frac{1}{1+5x^2}dx\ u = \sqrt{5}x$

   m) $\int \frac{(1+\sqrt{x})^{3/2}}{\sqrt{x}}dx,\ u = \sqrt{x}+1$ n) $\int \frac{x^3}{\sqrt{x^2+4}}dx,\ u = x^2+4$ o) $\int_{0}^{5} \frac{x+5}{\sqrt{x+3}}dx,\ u = x+3$

   p) $\int \frac{1}{\sqrt{1-(x-3)^2}}dx,\ u = x-3$ q) $\int \frac{x^2+2}{x+2}dx,\ u = x+2$ r) $\int \frac{\tan x}{\sec^3 x}dx,\ u = \sec x$

2. *Since* $x^2 + 4x + 7 = 3 + (x+2)^2$, $\int \frac{1}{x^2+4x+7} = \int \frac{1}{3+(x+2)^2}dx$*. The substitution* $\sqrt{3}u = x+2$ *gives*

   $$\int \frac{1}{x^2+4x+7}dx = \tfrac{\sqrt{3}}{3}\int \frac{1}{1+u^2}du = \tfrac{1}{\sqrt{3}}\arctan u + c = \tfrac{1}{\sqrt{3}}\arctan\left(\tfrac{x+2}{\sqrt{3}}\right) + c$$

   Integrate the following in a similar way:

   a) $\int \frac{1}{2+2x+x^2}dx$ b) $\int \frac{1}{x^2+4x+8}dx$ c) $\int \frac{1}{x^2+3x+10}dx$

3. *Since* $4x - x^2 - 3 = 1 - (x-2)^2$, $\int \frac{1}{\sqrt{4x-x^2-3}}dx = \int \frac{1}{\sqrt{1-(x-2)^2}}dx$*. Let* $u = x-2$*. Then*

   $$\int \frac{1}{\sqrt{4x-x^2-3}}dx = \int \frac{1}{\sqrt{1-u^2}}du = \arcsin u + c = \arcsin(x-2) + c.$$

   Integrate the following in a similar way

   (a) $\int \frac{1}{\sqrt{6x-x^2-8}}dx$ *(b)* $\int \frac{1}{\sqrt{8-2x-x^2}}dx$

4. *Evaluate the following: (a)* $\int_{3}^{11} \frac{x+1}{\sqrt{2x+3}}dx$ *(b)* $\int_{0}^{1} \frac{x^3}{x^2+1}dx$

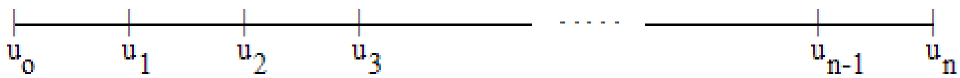
Justifying the Substitution Formula

Let $f(x)$ be a given function, and assume that we wish to determine $\int_a^b f(x)dx$. Let g be a differentiable function that maps some interval $[c,d]$ onto $[a,b]$. To simplify the argument, assume that g is increasing on $[c,d]$. In theory, to determine $\int_a^b f(x)dx$, we do the following:

- Select points $a = x_0 < x_1 < \cdots < x_n = b$ that divide $[a,b]$ into smaller subintervals $[x_0,x_1]$, $[x_1,x_2]$, $\ldots$, $[x_{n-1},x_n]$.
- Pick points t_1 in $[x_0,x_1]$, t_2 in $[x_1,x_2]$, $\ldots$, and t_n in $[x_{n-1},x_n]$, and
- Form Riemann sums $\sum_{i=1}^n f(t_i)\,(x_i - x_{i-1})$.

Then $\int_a^b f(x)dx$ is the limit of the above Riemann sums as the lengths of the subintervals shrink to 0.

Take points $c = u_0 < u_1 < \cdots < u_n = d$ in $[c,d]$, shown on the number line below,



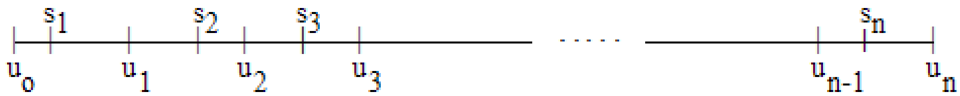
that divide $[c,d]$ into smaller subintervals $[u_0,u_1]$, $[u_1,u_2]$, $\ldots$, $[u_{n-1},u_n]$. We use them to select the points x_0, x_1, $\ldots$, x_n and t_0, t_1, $\ldots$, t_n in $[a,b]$, in the following special way: We apply the Mean Value Theorem to g on the interval $[u_0,u_1]$ to deduce that there is a number s_1 in the interval $[u_0,u_1]$, (see the figure below), such that

$$g(u_1) - g(u_0) = (u_1 - u_0)\,g'(s_1)$$

We then choose $x_0 = g(u_0)$, $x_1 = g(u_1)$ and $t_1 = g(s_1)$. We next apply the theorem to g on $[u_1,u_2]$ to obtain a number s_2 in the interval $[u_1,u_2]$ such that

$$g(u_2) - g(u_1) = (u_2 - u_1)\,g'(s_2),$$

then choose $x_2 = g(u_2)$ and $t_2 = g(s_2)$.



We continue in the same way till the nth step when we use the theorem to get a number s_n in the interval $[u_{n-1},u_n]$ such that

$$g(u_n) - g(u_{n-1}) = (u_n - u_{n-1})\,g'(s_n),$$

then choose $x_n = g(u_n)$ and $t_n = g(s_n)$. Now the above Riemann sum may be written as

$$\sum_{i=1}^n f(t_i)\,(x_i - x_{i-1}) = \sum_{i=1}^n f(g(s_i))\,[g(u_i) - g(u_{i-1})] = \sum_{i=1}^n f(g(s_i))g'(s_i)\,(u_i - u_{i-1})\,.$$

Clearly, $\sum_{i=1}^n f(g(s_i))g'(s_i)\,(u_i - u_{i-1})$ is a Riemann sum of $f(g(u))g'(u)$ on the interval $[c,d]$. The limit of such sums as the lengths of the intervals $[u_0,u_1]$, $[u_1,u_2]$, $\ldots$, $[u_{n-1},u_n]$ shrink to zero is $\int_c^d f(g(u))g'(u)du$. Therefore

$$\int_a^b f(x)dx = \lim \sum_{i=1}^n f(g(s_i))g'(s_i)\,(u_i - u_{i-1}) = \int_c^d f(g(u))g'(u)du.$$

More Practice Problems

1. Use the suggested substitution to evaluate the given definite integral:

a) $\displaystyle\int_1^4 \frac{3dx}{\sqrt{x}\,(2\sqrt{x}+1)^2} \quad u = 2\sqrt{x}+1.$

b) $\displaystyle\int_0^1 \frac{x^3dx}{x^2+1} \quad u = x^2+1.$

c) $\displaystyle\int_0^{\pi/4} \frac{\sec^2 xdx}{\sqrt{4+5\tan x}} \quad u = 4+5\tan x.$

d) $\displaystyle\int_0^4 x\left(\sqrt{3x+4}\right)dx \quad u = 3x+4.$

e) $\displaystyle\int_0^{\pi/4} \left(\frac{\cos x - \sin x}{\sin x + \cos x}\right)dx \quad u = \sin x + \cos x$

f) $\displaystyle\int_0^1 \frac{x^3dx}{3x^2+5} \quad u = 3x^2+5$

g) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\csc^2 x \cot xdx}{2+\csc x} \quad u = 2+\csc x.$

h) $\displaystyle\int_0^1 \frac{e^{2x}dx}{(1+e^x)^2} \quad u = e^x+1.$

i) $\displaystyle\int_1^4 \frac{\sec^4 xdx}{\sqrt{1+\tan x}} \quad u = 1+\tan x.$

j) $\displaystyle\int_1^{10} x^2\left(\sqrt{2x-1}\right)dx \quad u = 2x-1.$

k) $\displaystyle\int_0^1 \frac{(x+2)^3\,dx}{x+1} \quad u = x+1.$

l) $\displaystyle\int_1^4 \left(\frac{\sqrt{3+\sqrt{x}}}{\sqrt{x}}\right)dx \quad u = 3+\sqrt{x}$

m) $\displaystyle\int_0^{\pi/2} \frac{\sin^3 xdx}{\cos x+3} \quad u = \cos x+3.$ You may need the identity $\sin^2 x = 1-\cos^2 x$

n) $\displaystyle\int_0^4 x^3\left(\sqrt{x^2+9}\right)dx \quad u = x^2+9.$

o) $\displaystyle\int_1^{\pi/4} \frac{\sec^2 x \tan xdx}{1+\tan x} \quad u = 1+\tan x.$

p) $\displaystyle\int_0^{\pi/3} \frac{\sin x \cos xdx}{\sqrt{2-\sin x}} \quad u = 2-\sin x.$

q) $\displaystyle\int_0^8 (2x-1)\left(\sqrt{2x+1}\right)dx \quad u = 2x+1.$

r) $\displaystyle\int_0^1 \left(\frac{e^x - e^{-e}}{e^x + e^{-x}}\right)dx \quad u = e^x + e^{-x}$

2. Use a suitable substitution to evaluate the definite integral:

1. $\displaystyle\int_0^{1/2} \frac{x^3}{(1-x^2)^{3/2}}dx$

2. $\displaystyle\int_1^3 \left(x^3\sqrt{x^2+3}\right)dx$

3. $\displaystyle\int_0^2 \left(\frac{x^2}{\sqrt{x^3+1}}\right)dx$

4. $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\tan x}{\sqrt{\sec x}}dx$

5. $\displaystyle\int_0^4 \left(\frac{x^2}{\sqrt{2x+1}}\right)dx$

6. $\displaystyle\int_3^6 (x+2)^2\left(\sqrt{x-2}\right)dx$

7. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\cot x)(\csc x)^{\frac{5}{2}}\,dx$

8. $\displaystyle\int_0^5 \left(x^2\sqrt{x+4}\right)dx$

9. $\displaystyle\int_0^1 \left(\frac{x^3}{x^2+4}\right)dx$