Integration by Inspection

Recall that determining antiderivatives is the reverse of differentiating functions. Thus given a function $h(x)$, we have to answer the question: "*what is the most general function f whose derivative is $h(x)$?*" Equivalently, we imagine a table similar to the one below, giving the derivatives of various functions. Then determining an antiderivative means reading the table from right to left and add a constant to what you get.

Function	Derivative
1. x^n	nx^{n-1}
2. $\sin x$	$\cos x$
3. $\cos x$	$-\sin x$
4. $\tan x$	$\sec^2 x$
5. $\csc x$	$-\csc x \cot x$
6. $\sec x$	$\sec x \tan x$
7. $\cot x$	$-\csc^2 x$
8. e^x	e^x
9. $\ln\|x\|$	$\frac{1}{x}$
10. $\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
11. $\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
12. $\arctan x$	$\frac{1}{1+x^2}$
13. $\sinh x$	$\cosh x$
14. $\cosh x$	$\sinh x$
15. $\tanh x$	$\text{sech}^2 x$
16. $\coth x$	$-\text{csch}^2 x$
17. $f(x) \pm g(x)$	$f'(x) \pm g'(x)$
18. $kf(x)$, k a constant	$kf'(x)$

(1)

For example, to determine $\int \sec x \tan x dx$ we look for $\sec x \tan x$ in the column for derivatives. We find it in formula number 6 paired with $\sec x$, therefore

$$\int \sec x \tan x dx = \sec x + c.$$

The reality though, is that it is impossible to construct a table containing the derivative of every conceivable differentiable function because there are infinitely many of them. In table (1), we listed the derivatives of the building blocks x^n, $\sin x$, etc, and two rules for calculating derivatives. As the examples below demonstrate, it is possible to determine integrals of a number of more complicated functions by using the integrals of these elementary functions and the listed rules.

Example 1 *Consider* $\int \left(2\sin x - \frac{4}{x}\right) dx$. *Even though* $h(x) = 2\sin x - \frac{4}{x}$ *is not listed in table (1), its elementary components* $2\sin x$ *and* $-\frac{4}{x}$ *are indirectly listed. Indeed formulas 3 and 18 imply that* $2\sin x$ *is* *the derivative of* $-2\cos x$ *while 9 and 18 imply that* $-\frac{4}{x}$ *is the derivative of* $-4\ln x$*. It follows from formula 17 that* $2\sin x - \frac{4}{x}$ *is the derivative of* $-2\cos x - 4\ln x$*, therefore*

$$\int \left(2\sin x - \frac{4}{x}\right) dx = -2\cos x - 4\ln x + c$$

Example 2 *To determine* $\int \left(x^2+3x-1\right)\sqrt{x}dx$*, remove parentheses to get*

$$\int \left(x^2+3x-1\right)\sqrt{x}dx = \int \left(x^{5/2}+3x^{3/2}-x^{1/2}\right)dx$$

Now formulas 1 and 18 in the table imply that $x^{5/2}$ *is the derivative of* $\frac{2}{7}x^{7/2}$, $3x^{3/2}$ *is the derivative of* $\frac{6}{5}x^{5/2}$ *and* $-x^{1/2}$ *is the derivative of* $-\frac{2}{3}x^{3/2}$*. Therefore*

$$\int \left(x^2+3x-1\right)\sqrt{x}dx = \tfrac{2}{7}x^{7/2} + \tfrac{6}{5}x^{5/2} - \tfrac{2}{3}x^{3/2} + c$$

Example 3 *Consider* $\int \frac{3x+1}{2\sqrt{x}}dx$*. Unlike derivatives, there is no quotient rule for integrals, therefore divide as much as possible then look for antiderivatives. The result is*

$$\int \left(\frac{3x}{2\sqrt{x}} + \frac{1}{2\sqrt{x}}\right)dx = \int \left(\tfrac{3}{2}\sqrt{x} + \frac{1}{2\sqrt{x}}\right)dx = \int \left(\tfrac{3}{2}x^{1/2} + \tfrac{1}{2}x^{-1/2}\right)dx.$$

Formula 1 of the table implies that $\frac{3}{2}\sqrt{x}$ *is the derivative of* $x^{3/2}$ *and* $\frac{1}{2}x^{-1/2}$ *is the derivative of* $x^{1/2}$*. Therefore*

$$\int \frac{3x+1}{2\sqrt{x}}dx = x^{3/2} + x^{1/2} + c$$

Example 4 *To determine* $\int x^2\cos x^3 dx$*, observe that the integrand is a product of the terms* $\cos x^3$ *and* x^2*. The chain rule suggests that it was obtained by differentiating an expression involving* $\sin x^3$*, because*

The Derivative of $\sin(\quad)$ *is* $\cos(\quad)\times$ *Derivative of what is in* $(\quad)$

Since the derivative of $\sin x^3$ *is* $\left(\cos x^3\right)\left(3x^2\right)$*, which is not quite* $x^2\cos x^3$*, the choice* $\sin x^3$ *is off the target by a* ***constant*** *3. But that is easy to fix; we simply divide* $\sin x^3$ *by 3. Therefore*

$$\int x^2\cos x^3 dx = \tfrac{1}{3}\sin x^3 + c$$

You can easily check that the derivative of $F(x) = \frac{1}{3}\sin x^3 + c$ *is* $F'(x) = x^2\cos x^3$.

Example 5 *To determine* $\int x\sqrt{3x^2+4}dx$*, we also note that the presence of the terms* $\sqrt{3x^2+4} = \left(3x^2+4\right)^{1/2}$ *and* x *suggest that the integrand must be the result of differentiating an expression involving* $\left(3x^2+4\right)^{3/2}$*. By the chain rule, the derivative of* $\left(3x^2+4\right)^{3/2}$ *is*

$$\tfrac{3}{2}\cdot\left(3x^2+4\right)^{1/2}(6x) = 9x\sqrt{3x^2+4}$$

which is off what we want by the constant factor 9. We fix this by dividing by 9. Therefore

$$\int x\sqrt{3x^2+4}dx = \tfrac{1}{9}\left(3x^2+4\right)^{3/2} + c$$

It should be easy to verify that the derivative of $f(x) = \frac{1}{9}\left(3x^2+4\right)^{3/2} + c$ *is* $x\sqrt{3x^2+4}$.

The Trial-And-Error method we used in the above examples is called **integration by inspection** (or integration by guessing wisely).

Exercise 6

1. *Integrate each function by inspection. Check your answer by differentiating your indefinite integral.*

(1) $\int \sqrt{x}dx$ (2) $\int x^{10}dx$ (3) $\int \frac{1}{x^4}dx$

(4) $\int x^{-1}dx = \int \frac{1}{x}dx$ (5) $\int x^n dx,\ n \neq -1$ (6) $\int \frac{3}{\sqrt{x}}dx$

(7) $\int e^{4x}dx$ (8) $\int e^{0.5x+6}dx$ (9) $\int e^{-x}dx$

(10) $\int \left(\frac{4}{x} - \frac{x}{4}\right) dx$ (11) $\int \left(\pi^2 + \frac{8}{x}\right) dx$ (12) $\int \left(\frac{2}{\sqrt{1-x^2}}\right) dx$

(13) $\int \frac{3}{x^2+1}dx$ (14) $\int \left(x^3 - \frac{1}{4x^3}\right) dx$ (15) $\int \left(\sqrt{x} - \sqrt{2}\right) dx$

(16) $\int 3\csc x \cot x dx$ (17) $\int \left(7x^{\frac{5}{2}} - x^{\frac{3}{2}}\right) dx$ (18) $\int \left(1 + x\sqrt{x}\right) dx$

(19) $\int \left(\pi x - \sec^2 x\right) dx$ (20) $\int \left(\frac{3x+4}{\sqrt{x}}\right) dx$ (21) $\int \left(\frac{x^2+1}{x^3}\right) dx$

(22) $\int \left(\frac{2}{\pi x^2} - 3\right) dx$ (23) $\int \left(2x+1\right)^2 dx$ (24) $\int x\left(3x^2+1\right)^3 dx$

(25) $\int x^2\left(4x^3+1\right)^4 dx$ (26) $\int x^3\left(5x^4+1\right)^5 dx$ (27) $\int \left(\sqrt{8x+1}\right) dx$

(28) $\int \left(x\sqrt{8x^2+1}\right) dx$ (29) $\int \left(x^2\sqrt{x^3+1}\right) dx$ (30) $\int \frac{1}{\sqrt{8x+1}}dx$

(31) $\int \frac{x}{\sqrt{8x^2+1}}dx$ (32) $\int \frac{x^2}{\sqrt{8x^3+1}}dx$ (33) $\int \left(2x-3\right)^8 dx$

(34) $\int x\left(2x^2-3\right)^8 dx$ (35) $\int x^2\left(\frac{x^3}{2} - 3\right)^8 dx$ (36) $\int x^3\left(2x^4-3\right)^8 dx$

(37) $\int \sin 5x dx$ (38) $\int \sin(5x-4)dx$ (39) $\int \sin(\frac{2}{3}x+1)dx$

(40) $\int \sin(ax+b)dx$ (41) $\int x\sec^2\left(x^2\right) dx$ (42) $\int x^2\sec^2\left(x^3\right) dx$

(43) $\int x^3\sec^2\left(x^4\right) dx$ (44) $\int x^4\sec^2\left(x^5\right) dx$ (45) $\int xe^{x^2}dx$

(46) $\int x^2e^{x^3}dx$ (47) $\int x^3e^{x^4}dx$ (48) $\int x^4e^{x^5}dx$

2. *(In this exercise, you obtain a result that we will soon use to derive Simpson's rule.) Let* $q(x) = Ax^2 + Bx + C$ *be a quadratic function and* $[a, b]$ *be an interval.*

   (a) *Show that* $\int_a^b q(x)dx = \frac{(b-a)}{6}\left[2A(b^2+ab+a^2) + 3B(a+b) + 6C\right]$. *(Hint:* $b^3 - a^3$ *factors as* $(b-a)\left(b^2+ab+a^2\right)$*.)*

   (b) *Show that* $2A(b^2+ab+a^2) + 3B(a+b) + 6C$ *may be written as*

   $$\left(Aa^2 + Ba + C\right) + \left(Ab^2 + Bb + C\right) + A(a+b)^2 + 2B(a+b) + 4C$$

   (c) *Note that* $A(a+b)^2 + 2B(a+b) + 4C = 4\left[A\left(\frac{a+b}{2}\right)^2 + B\left(\frac{a+b}{2}\right) + C\right]$. *Use this to deduce that* $\int_a^b q(x)dx = \frac{(b-a)}{6}\left[q(a) + 4q\left(\frac{a+b}{2}\right) + q(b)\right]$.