The Derivative of a Function

As you will soon find out, slopes of tangents are crucial in solving a variety of problems. Therefore, you should not be surprised that given a function f, it is natural to ask for an expression for the slope of the tangent at an arbitrary point $(x, f(x))$ on its graph. We use this expression to introduce a new function called the derivative of f . More precisely:

Definition 1 *The derivative of a given function f is another function, denoted by f' or $\frac{df}{dx}$ whose value at a number x is the slope of the tangent to the graph of f at $(x, f(x))$.*

Example 2 *We showed that the slope of the tangent to the graph of $g(x) = x^2$ at $\left(x, x^2\right)$ is $2x$. It follows that the derivative of g is the function g' with formula $g'(x) = 2x$. Alternatively, $\frac{dg}{dx} = 2x$.*

Example 3 *We showed that the slope of the tangent to the graph of $w(x) = \sqrt{x}$ at $(x, \sqrt{x})$, $x > 0$, is $\frac{1}{2\sqrt{x}}$. Therefore the derivative of w is the function w' with formula $w'(x) = \frac{1}{2\sqrt{x}}$. Alternatively, $\frac{dw}{dx} = \frac{1}{2\sqrt{x}}$.*

Determining the derivative of $f(x)$ is called differentiating f with respect to x. The independent variable does not have to be x. If it is another letter, say t, then determining the derivative of $f(t)$ is called differentiating f with respect to t. Since $f'(x)$ is easier to write than $\frac{df}{dx}$, we will mostly use f' for the derivative until we get to problems in which the latter notation is more convenient.

A function that has a derivative is called a differentiable function.

Example 4 *We noted earlier, (and promised a proof in the exercises to come), that if r is any real number then the slope of the tangent to the graph of $f(x) = x^r$ at (x, x^r) is rx^{r-1}. Therefore the derivative of $f(x) = x^r$ is the function f' with formula*

$$f'(x) = rx^{r-1}.$$

*This is called the **power rule for derivatives**.*

Example 5 *Let $f(x) = x\sqrt{x}$. We may re-write as a single exponent: $f(x) = x^{3/2}$. Then by the power rule for derivatives, $f'(x) = \frac{3}{2}x^{1/2}$.*

Example 6 *Let $f(x) = \frac{1}{\sqrt[5]{x^7}}$. We may re-write it as an exponent: $f(x) = \frac{1}{x^{7/5}} = x^{-7/5}$. Then $f'(x) = -\frac{7}{5}x^{-12/5}$*

Example 7 *Let $f(x) = mx$. Its graph is a straight line ℓ with slope m. The tangent to ℓ at any point (x, mx) is the line ℓ itself, therefore its slope is m. It follows that $f'(x) = m$. More precisely, if m is a constant then the derivative of $f(x) = mx$ is $f'(x) = m$.*

Example 8 *Let f be a constant function. Thus f has formula $f(x) = c$ where c is a constant. We also pointed out that the slope of the tangent at any point (x, c) on the graph is 0. Therefore, if $f(x) = c$, (a constant), then $f'(x) = 0$.*

Formal Definition of a Derivative

Let f be a given function and $(x, f(x))$ be a point on its graph. To calculate the slope of the tangent to the graph of f at $(x, f(x))$, we evaluate the quotient

$$\frac{[f(x+h) - f(x)]}{h}$$

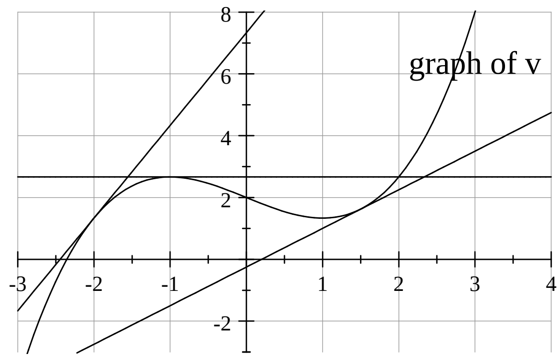
then ask the question: *is there a single number that is close to all such quotients when h is close to 0?* If there is, then it is called the limit of $\frac{[f(x+h) - f(x)]}{h}$ as h approaches 0 and it is defined to be the slope of the tangent at $(x, f(x))$. Therefore:

Definition 9 *The derivative of a given function f is the function f' with formula*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{1}$$

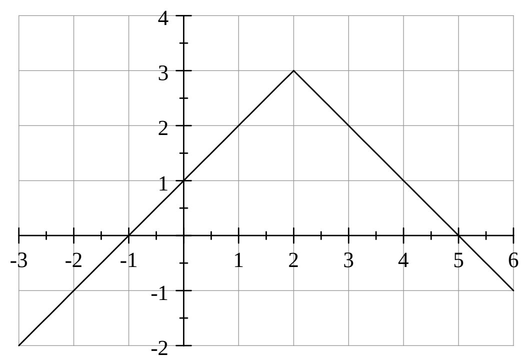
Exercise 10

1. *The graph of some function v is given below together with the tangents to the graph at the three points with x-coordinates -2, -1, and 1.5 respectively.*



   Use the tangents to estimate $v'(-2)$, $v'(-1)$ and $v'(1.5)$.

2. *The graph of some function u is given below.*



   Determine $u'(a)$ when $a < 2$ and $u'(b)$ when $b > 2$. (Since one cannot draw a tangent at $(2,3)$, $u'(2)$ is undefined.)

3. *Use the definition of a derivative as a limit of a quotient to show that the derivative of $f(x) = mx + b$ is $f'(x) = m$ and the derivative of $g(x) = c$, (a constant), is $g'(x) = 0$.*

4. *Let $f(x) = \dfrac{1}{x^2}$, $x \neq 0$. Show that $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-2x - h}{x^2\,(x+h)^2}$ then determine $f'(x)$.*

5. *Let $g(x) = \dfrac{1}{\sqrt{x}}$, $x > 0$. Show that $\dfrac{g(x+h) - g(x)}{h} = \dfrac{\sqrt{x} - \sqrt{x+h}}{h\left(\sqrt{x(x+h)}\right)}$. Now rationalize the numerator and determine $g'(x)$.*

6. *Let $u(x) = x^3 + x$. Show that $\dfrac{u(x+h) - u(x)}{h} = 3x^2 + 1 + 3xh + h^2$ then determine $u'(x)$.*

7. *Determine $f'(x)$ given that $f(x) =$*

(a) x^{23}	*(b)* x^{-3}	*(c)* $\frac{1}{x^5}$	*(d)* $x^{3/4}$	*(e)* 100.5
(f) $\sqrt[4]{x^7}$	*(g)* $\frac{1}{x^{4/5}}$	*(h)* $x^2\sqrt{x}$	*(i)* $\frac{1}{x^{5/2}}$	*(j)* $\frac{1}{\sqrt[2]{x^7}}$
(k) $\sqrt[3]{x^5}$	*(l)* $x^{\sqrt{3}}$	*(m)* π^3	*(n)* $x^3\sqrt{x}$	*(o)* $\frac{1}{x^3\sqrt{x}}$