Complex Numbers

If you have not yet encountered complex numbers, you will soon do so in the process of solving quadratic equations. The general quadratic equation $Ax^2 + Bx + C = 0$ has solutions

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad \text{or} \quad x = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

For example, $3x^2 + 2x - 1 = 0$ has solutions

$$x = \frac{-2 + \sqrt{4 - 4\,(3)\,(-1)}}{2\,(3)} \quad \text{or} \quad x = \frac{-2 - \sqrt{4 - 4\,(3)\,(-1)}}{2\,(3)}$$

These expressions simplify to $x = \dfrac{1}{3}$ or $x = -1$

When we apply the same formula to $x^2 + 2x + 5 = 0$ we arrive at

$$x = \frac{-2 \pm \sqrt{4 - 4\,(1)\,(5)}}{2\,(1)}$$

These simplify to

$$x = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm \sqrt{16}\sqrt{-1}}{2} = \frac{-2 \pm 4\sqrt{-1}}{2}$$

but we cannot go further because there is no real number whose square is -1, (since the square of every non-zero real number is positive). The only way forward is to introduce more numbers. We take the bold step of introducing a new number, to be denoted by i, with the property that $i^2 = -1$. Then

$$\sqrt{-1} = \pm i$$

Now we can solve the quadratic equation $x^2 + 2x + 5 = 0$ to get

$$x = \frac{-2 + 4i}{2} \quad \text{or} \quad x = \frac{-2 - 4i}{2}$$

These may be simplified as follows: $\dfrac{-2 + 4i}{2} = \dfrac{2(-1 + 2i)}{2} = -1 + 2i$ and $\dfrac{-2 - 4i}{2} = \dfrac{2(-1 - 2i)}{2} = -1 - 2i$ respectively.

It turns out that introducing this new number enables us to solve any given quadratic equation.

Examples

(a) *The quadratic equation* $x^2 - 4x + 13 = 0$ *has solutions*

$$x = \frac{4 + \sqrt{16 - 4(1)(13)}}{2} = \frac{4 \pm 6i}{2}$$

These may be simplified to $2 + 3i$ *or* $2 - 3i$ *respectively.*

(b) The quadratic equation $2x^2 - 3x + 7 = 0$ *has solutions*

$$x = \frac{3 + \sqrt{9 - 56}}{2} = \frac{3 \pm \sqrt{47}i}{2}$$

Since 47 *is not a perfect square, we leave the expressions* $\dfrac{3 + \sqrt{47}i}{2}$ *and* $\dfrac{3 - \sqrt{47}i}{2}$ *as they are.*

Exercise 1 *Solve the following quadratic equations:*

1. $x^2 - 4x + 8 = 0$

2. $x^2 + 2x + 10 = 0$

3. $2x^2 + 7x + 6 = 0$

4. $3x^2 - x + 2 = 0$

Expressions like $2+3i$, $-1-2i$, $\dfrac{3-\sqrt{47}i}{2} = \dfrac{3}{2} - \dfrac{\sqrt{47}}{2}i$ are called complex numbers. (The term "complex numbers" should not intimidate you. There is nothing complex about them.) They all have the form $a+bi$, *called the standard form*, where a and b are real numbers and i is the new number we have introduced with the property that $i^2 = -1$.

Some Standard Terms

Given a complex number $a + bi$, the number a that is not multiplied by i is called the real part of $a + bi$. The number b that is multiplied by i is called the complex part of $a + bi$. Thus the real part of $-1 - 2i$ is -1 and its complex part is -2. In the case of $\dfrac{3}{2} - \dfrac{\sqrt{47}}{2}i$, the real part is $\dfrac{3}{2}$ and the complex part is $-\dfrac{\sqrt{47}}{2}$.

A number like $-3i$ which has no real part is called a purely complex number. Likewise, a number like 6 which has no complex part is called a purely real number.

Manipulating complex numbers

Adding/subtracting complex numbers

To add complex numbers, *simply add the real parts then and do the same thing to the complex parts.* Thus

$$(3+2i) + (5-6i) = (3+2) + (2-6)\,i = 5 - 4i \qquad \text{and} \qquad (-5+2i) + (-1-9i) + 6i = -6 - i$$

To subtract $(c + di)$ from $a + bi$, you do what you would expect: *Subtract c from a, to get the real part of the difference, then subtract d from b to get the complex part of the difference.* In other words

$$(a + bi) - (c + di) = (a - c) + (b - d)\,i.$$

Thus

$$(2 - 3i) - (4 - 5i) = -2 + i \qquad \text{and} \qquad 5i - (3 + 2i) - (1 - 4i) = -4 + 7i$$

Exercise 2

1. *Carry out the following operations:*

   (i) $3 - 7i + (4 - 5i)$ *(ii)* $(4 - 3i) + \left(\frac{2}{3} + \frac{1}{2}i\right)$ *(iii)* $2 - 7i + (6 - 5i) + (-1 + 4i)$

2. *Find the sum of the two solutions of the quadratic equation* $x^2 + 4x + 8 = 0$

3. *Carry out the following operations:*

   (i) $5 - 6i - (4 - 3i)$ *(ii)* $(2 - 3i) - \left(\frac{3}{2} + \frac{1}{3}i\right)$ *(iii)* $2 - 7i - (6 - 5i) - (-1 + 4i)$

4. *Determine the two solutions of the quadratic equation* $4x^2 + 4x + 5 = 0$ *then subtract one of them from the other.*

5. *Show that if you add a complex number to its complex conjugate then the result must be a purely real number. What is the result of subtracting a complex number from its complex conjugate?*

6. *Show that if the solutions of a quadratic equation* $Ax^2 + Bx + C = 0$ *are complex then one of them is a complex conjugate of the other.*

Multiplying complex numbers

To multiply $a+bi$ and $c+di$, simply multiply as usual then collect like terms. But along the way use the fact that $i^2=-1$. Examples:

$$(3-2i)(2+4i)=6+12i-4i-8i^2=6+8i-8(-1)=14+8i$$

$$(2-i)(5-3i)=10-6i-5i+3i^2=10-11i+3(-1)=7-11i$$

$$(3-2i)(3+2i)=9-6i+6i-4i^2=9-4(-1)=13$$

$$\left(1+\tfrac{2i}{3}\right)\left(1-\tfrac{2i}{3}\right)=1-\tfrac{2i}{3}+\tfrac{2i}{3}-\tfrac{4}{9}(-1)\,1+\tfrac{4}{9}=\tfrac{13}{9}$$

Exercise 3

1. *Do the multiplication and write the result in the standard form* $a+bi$.

   (i) $(3-4i)(5-6i)$ *(ii)* $(3+5i)(3-5i)$ *(iii)* $5i(1+3i)(2+3i)$

2. *Find the product of the two solutions of the quadratic equation* $4x^2-4x+5=0$

3. *Note that* $i^2=-1$, $i^3=i^2i=-i$, $i^4=i^2i^2=(-1)(-1)=1$, $i^5=i^4i=i$, ... *Use these observations to put the following phrases in their correct positions below. The phrases are (a) n is divisible by 4, (b) n leaves remainder 1 when it is divided by 4, (c) n leaves remainder 2 when it is divided by 4, (d) n leaves remainder 3 when it is divided by 4.*

$$i^n=\begin{cases} 1 & \textit{if} \\ -1 & \textit{if} \\ i & \textit{if} \\ -i & \textit{if} \end{cases}$$

Dividing by a complex number

Before doing the division, we have to introduce the concept of a complex conjugate. Here are three examples:

The complex conjugate of $2+3i$ is the number $2-3i$.

The complex conjugate of $7-5i$ is the number $7+5i$.

The complex conjugate of $-4-9i$ is the number $-4+9i$.

In general, the complex conjugate of a given complex number is obtained by simply changing the sign of the complex part of the given number. If it is positive, you change it to a negative. If it is negative, you change it to a positive. In the table below, we give the complex conjugate of each given number then form the product of the two numbers, (i.e. the product of the number and its conjugate). Remember that $i^2=-1$.

Number	Its conjugate	The product of the number and its conjugate
$5-2i$	$5+2i$	$(5-2i)(5+2i)=5^2+10i-10i+2^2=5^2+2^2$
$-1+\frac{1}{3}i$	$-1-\frac{1}{3}i$	$\left(-1+\frac{1}{3}i\right)\left(-1-\frac{1}{3}i\right)=(-1)^2+\frac{1}{3}i-\frac{1}{3}i+\left(\frac{1}{3}\right)^2=(-1)^2+\left(\frac{1}{3}\right)^2$
$\frac{4}{3}+\frac{2}{3}i$	$\frac{4}{3}-\frac{2}{3}i$	$\left(\frac{4}{3}+\frac{2}{3}i\right)\left(\frac{4}{3}-\frac{2}{3}i\right)=\left(\frac{4}{3}\right)^2-\frac{8}{9}i+\frac{8}{9}i+\left(\frac{2}{3}\right)^2=\left(\frac{4}{3}\right)^2+\left(\frac{2}{3}\right)^2$
$a+bi$	$a-bi$	$(a+bi)(a-bi)=a^2-abi+abi+b^2=a^2+b^2$
$u-iv$	$u+iv$	$(u-iv)(u+iv)=u^2+uvi-uvi+v^2=u^2+v^2$

The main point to take away from the above table is that the product of a number and its complex conjugate is a *purely real number*. This is the property we use to move a complex number from the denominator to the numerator in a given quotient.

Example 4 *To divide $2+4i$ by $1-3i$ and write the result in the standard form $a+bi$ of a complex number. We start with the expression*

$$\frac{2+4i}{1-3i}$$

and proceed to remove the complex number from the denominator. As suggested by the above observations regarding a complex number and its conjugate, simply multiply the denominator of the above fraction by the complex conjugate of $1-3i$, which is $1+3i$. But we must avoid changing the given fraction, therefore we must also multiply the numerator by the same complex conjugate. The result is

$$\begin{aligned}\frac{2+4i}{1-3i} &= \frac{(2+4i)(1+3i)}{(1-3i)(1+3i)} = \frac{2+6i+4i+12i^2}{1+3i-3i-9i^2} = \frac{2+10i+12(-1)}{1-9(-1)} = \frac{-10+10i}{10} \\ &= \frac{10(-1+i)}{10} = -1+i\end{aligned}$$

Example 5 *To divide $7-2i$ by $3i$. Thus we want to write $\frac{7-2i}{3i}$ in the standard form $a+bi$ of a complex number. We multiply the numerator and denominator of $\frac{7-2i}{3i}$ by the conjugate of $3i$, (which is $-3i$), then simplify. The result is*

$$\frac{7-2i}{3i} = \frac{(7-2i)(-3i)}{(3i)(-3i)} = \frac{-21i-6}{9} = -\frac{2}{3} - \frac{7i}{3}$$

Example 6 *To write $\frac{i}{3+i} - \frac{2}{1-2i}$ in the standard form $a+bi$. One way to do this is to remove the complex numbers from both numerators. Thus multiply the numerator and denominator of $\frac{i}{3+i}$ by $(3-i)$ and then multiply the numerator and denominator of $\frac{2}{1-2i}$ by $(1+2i)$. The result is*

$$\begin{aligned}\frac{i}{3+i} - \frac{2}{1-2i} &= \frac{i(3-i)}{(3+i)(3-i)} - \frac{2(1+2i)}{(1-2i)(1+2i)} = \frac{3i+1}{10} - \frac{2+4i}{5} \\ &= \frac{1}{10} - \frac{2}{5} + \frac{3i}{10} - \frac{4i}{5} = -\frac{3}{10} - \frac{1}{2}i\end{aligned}$$

Exercise 7

1. *Show that* $\frac{6-4i}{1+i} = 1-5i$
2. *Divide* $3+5i$ *by* $1-3i$.
3. *Perform the operations:*

(a) $2-3i+(a-4i)$	*(b)* $(4-2i)-\left(2+\frac{3}{4}i\right)$	*(c)* $(3-5i)+(-2+7i)-(4-i)$
(d) $i(4-i)$	*(e)* $(6+7i)(5-4i)$	*(f)* $(1+i)(1-2i)(2+3i)$
(g) $\frac{5-10i}{1+2i}$	*(h)* $\frac{26-13i}{2+3i}$	*(i)* $\frac{10i}{3-i}$
(j) $\frac{2}{1-i} - \frac{3}{1+i}$	*(k)* $\frac{3}{i} - \frac{2+3i}{1+2i}$	*(l)* $\frac{2}{2+i} - 4 + i$

4. *You are given the quadratic equation* $x^2 + 6x + 10 = 0$

   (a) *Determine its two solutions.*

   (b) *Show that when you multiply the two solutions the result is 10.*

5. *You are given the quadratic equation* $x^2 + bx + c$ *where* b *and* c *are constants.*

   (a) *Determine its two solutions.*

   (b) *Show that when you add the two solutions, the result is* $-b$ *and when you multiply the two solutions the result is* c.

6. *Write the following in standard form*

   a. $\dfrac{1+i}{1-i}$ b. $\dfrac{3-3i}{3+3i}$ c. $\dfrac{a+ai}{a-ai}$ d. $\dfrac{b-bi}{b+bi}$

7. *Show that if the real part of a complex number* z *is equal to its complex part then the real part of* z^2 *is zero. What is the complex part of* z^2 *equal to?*

8. *The real part of a complex number* z *is the negative of its complex part. What can you conclude about: (a) the real part of* z^2*? (b) the complex part of* z^2*?*