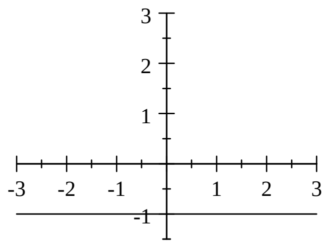
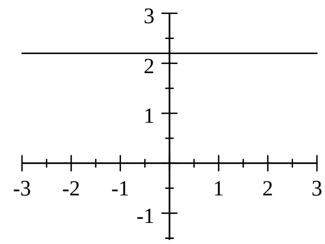
Calculating Slopes of Tangents

The maximization problems we addressed show that slopes of tangents are usefull problem-solving tools. In this section, we systematically compute the slopes of tangents to graphs of some familiar functions.

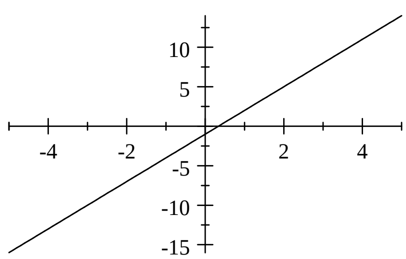
We start with the "linear functions", (i.e. the functions whose graphs are straight lines). The simplest are the constant ones like $g(x) = -1$ and $h(x) = 2.2$ whose graphs are shown below. In general, if c is a



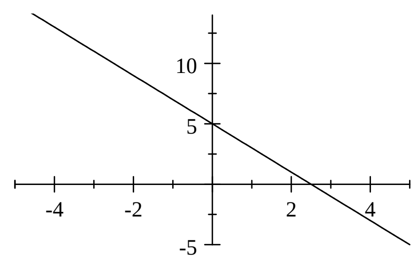


constant then the graph of $f(x) = c$ is a horizontal straight line. The tangent to its graph at any point (x, c) is the horizontal line itself. Since a horizontal line has zero slope, the slope of the tangent to the graph of $f(x) = c$ at any point (x, c) is 0.

In general, a linear function has a formula $f(x) = mx + c$ where m and c are constants, and its graph is a straight line ℓ with slope m. The tangent to ℓ at any point $(x, f(x))$ is the line ℓ itself. Therefore the slope of the tangent at any point on the graph of $f(x) = mx + c$ is m. For example, the slopes of the tangents to the graphs of $g(x) = 3x - 1$ and $h(x) = -2x + 5$, (shown below), are 3 and -2 respectively.

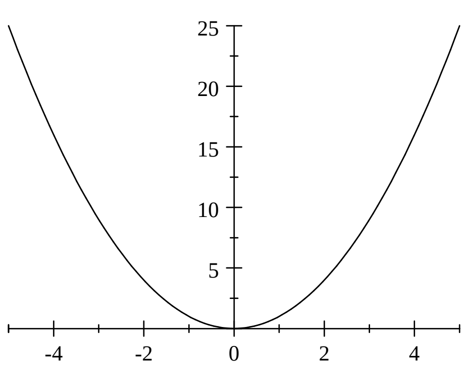


Graph of $g(x) = 3x - 1$

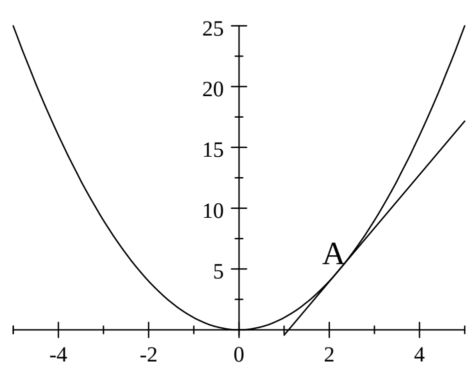


Graph of $h(x) = -2x + 5$

After the linear functions, the quadratic functions like $f(x) = x^2$ should be next. The graph of f is a parabola shown below. We wish to determine the slope m of the tangent to the graph at an arbitrary point $A\left(a, a^2\right)$. The second figure shows such a tangent.



Graph of $f(x) = x^2$

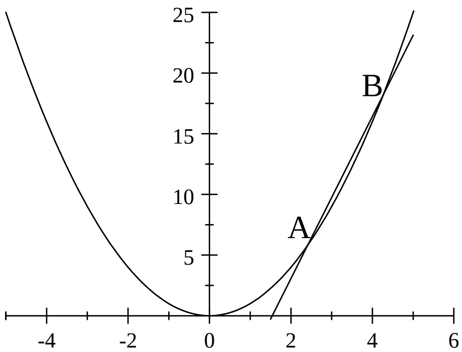


Graph of $f(x) = x^2$ *and tangent at* A

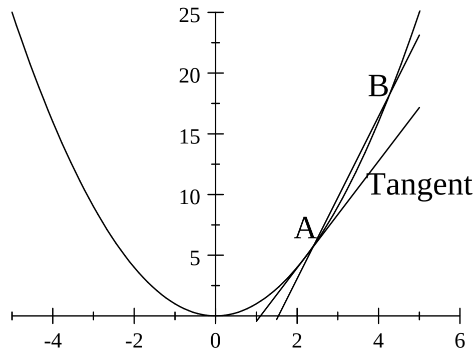
As pointed out in the introduction, the standard method of calculating the slope of a line using the formula

$$Slope\ of\ line = \frac{Rise}{Run}$$

cannot be used here because we know only one point on the line, namely $\left(a, a^2\right)$, and we need two in order to calculate a *Rise* and corresponding *Run.* We therefore turn to the next best step, which is to calculate approximate values of m then identify the single number that is close to all the *good approximations.* We use **secant lines** through $\left(a, a^2\right)$ to obtain these approximations. As pointed out in the introduction, a secant line through $\left(a, a^2\right)$ is a line segment that passes through $\left(a, a^2\right)$ and another point on the graph of f. An example is the line segment through $A\left(a, a^2\right)$ and $B\left(a+2, (a+2)^2\right)$ shown in the figure below. It is drawn together with the tangent at A in the second figure.



Secant line joining A and B

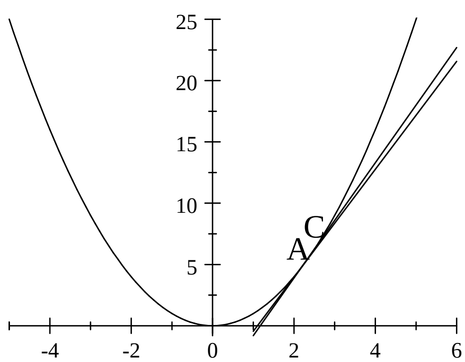


The *secant line and the tangent at A*

Its slope, can be calculated because we know two of its points, (namely A and B), and it is equal to

$$\frac{(a+2)^2 - a^2}{(a+2) - a} = 2a + 2.$$

Therefore $m \simeq 2a + 2$. This is not really a good approximation because, (as the second figure above shows), AB does not approximate the tangent that well. We should get a better one by taking a secant line joining $\left(a, a^2\right)$ to a closer point than $(a+2, (a+2)^2)$. An example is the secant line joining $\left(a, a^2\right)$ and $C = (a+0.2, f(a+0.2)) = \left(a+0.2, a^2+0.4a+0.04\right)$, drawn in the figure below.



Its slope is $\dfrac{f(a+0.2) - f(a))}{a+0.2-a} = \dfrac{0.4a+0.04}{0.2} = 2a + 0.2$, therefore $m \simeq 2a + 0.2$

For an even better approximation, take the line joining $\left(a, a^2\right)$ and $(a+0.01, f(a+0.01))$. Its slope is

$$\frac{f(a+0.01) - f(a)}{a+0.01-a} = \frac{0.02a+0.0001}{0.01} = 2a + 0.01.$$

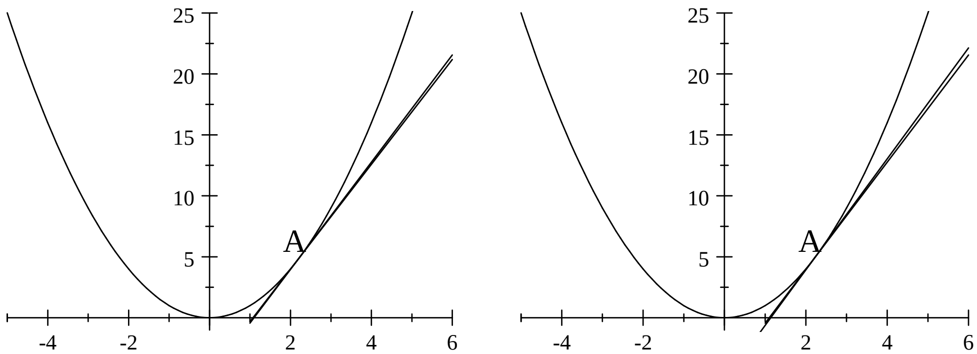
We now generalize as follows: Take a *general* point P close to $\left(a, a^2\right)$. It is obtained by adding a small variable h, (which may be positive or negative), to a then evaluate $f(a+h)$. Thus, P has coordinates

$$(a+h, f(a+h)) = \left(a+h, a^2+2ah+h^2\right).$$

The slope of the secant line joining A and P is

$$\frac{f(a+h)-f(a)}{h} = \frac{a^2+2ah+h^2-a^2}{h} = \frac{h(2a+h)}{h} = 2a+h \tag{1}$$

The smaller h is in absolute value, the closer the secant line joining $\left(a, a^2\right)$ to $(a+h, f(a+h))$ is to the tangent line. The tangents and corresponding secant lines when $h=-0.1$ and $h=0.15$ are shown in the figures below.



Tangent and secant line when $h=-0.1$ *Tangent and secant line when $h=0.15$*

It follows that the *good approximations* to the slope of the tangent are the numbers $2a+h$ where h is close to 0. Therefore the single number that is close to all the *good approximations to the slope of the tangent* is $2a$ and this must be the slope of the tangent at $\left(a, a^2\right)$

For a quadratic function like this, it is possible to double check this result using an algebraic argument as follows: If the slope of the tangent is m, (it is unknown at this moment), then its equation is $\left(y-a^2\right) = m(x-a)$, which we may also write as $y = mx - ma + a^2$. The equation that gives the point(s) of intersection of the graph of f and the line $y = mx - ma + a^2$ is $x^2 = mx - ma + a^2$. In the standard form of a quadratic equation, it is

$$x^2 - mx + ma - a^2 = 0 \tag{2}$$

Since a tangent to a parabola intersects the parabola in only one point, equation (2) above has only one solution. (This is the condition that enables us to determine m in the case of a parabola. It may not hold for other curves.) Recall that for a quadratic equation $Ax^2+Bx+C=0$ to have one root, its discriminant B^2-4AC must be 0. In the case of (2), $A=1$, $B=m$ and $C=ma-a^2$, therefore m must satisfy the equation

$$m^2 - 4(ma - a^2) = 0 \tag{3}$$

When we remove the parentheses, (3) becomes $m^2 - 4am + 4a^2 = 0$ which factors as $(m-2a)^2 = 0$. Therefore $m = 2a$, the very value we deduced from approximate values.

Returning to the ratio

$$\frac{f(a+h)-f(a)}{h} = 2a+h,$$

we noted that if you substitute numbers h close to 0 into this expression, then $2a$ is the number that is close to the answers you get. Because of this, we say that

"the limit of $\frac{\mathbf{f(a+h)-f(a)}}{\mathbf{h}}$ as h approaches 0 is $2a$"

This is written briefly as

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=2a$$

Remember that this is just a formal way of stating that if you substitute any number h that is sufficiently close to 0 into the expression $\dfrac{f(a+h)-f(a)}{h}$, you get a value close to $2a$.

Remark 1 *We have essentially established that the slope of the tangent to the graph of $f(x)=x^2$ at any point $\left(x,x^2\right)$ is $2x$. We may use this result to answer a number of specific questions about the graph of f. For example*

- *What is the slope of the tangent at $(2.5, 6.25)$?* And the answer is $2\times(2.5)=5$.
- *What is the equation of the tangent at $(-3,9)$?* Answer: Its slope is $2\times(-3)=-6$, therefore its equation is
$$(y-9)=-6\,(x+3)\qquad\text{or}\qquad y=-6x-9.$$
- *At what point on the graph of g is the slope of the tangent equal to 7?* And the answer is $\left(3.5,(3.5)^2\right)=(3.5,12.25)$ since 3.5 is the solution to the equation $2x=7$.
- *Where, on the graph of f, is the tangent parallel to the line $3y+4x=2$?* Answer; at the point where the slope of the tangent is $-\frac{4}{3}$. The solution to the equation $2x=-\frac{4}{3}$ is $x=-\frac{2}{3}$. Therefore the point is $(-\frac{4}{3},\frac{4}{9})$.
- *For what values of x is the slope of the tangent at $\left(x,x^2\right)$ negative?* Answer; any $x<0$

The next function to consider is $u(x)=x^3$. It turns out that the slope of the tangent at any point $A\left(x,x^3\right)$ on its graph is $3x^2$. To see this, take a point $P(x+h,(x+h)^3)$ near A. The slope of the secant line joining A and P is

$$\frac{u(x+h)-u(x)}{h}=\frac{3x^2h+3xh^2+h^3}{h}=\frac{h\left(3x^2+3xh+h^2\right)}{h}=3x^2+h(3x+h)$$

The slope of the tangent is the single number that is close to all the values $3x^2+h(3x+h)$ when h is close to 0. More precisely, it is the limit of $3x^2+h(3x+h)$ as h approaches 0. That number is

$$3x^2+0(3x+0)=3x^2,$$

therefore, the slope of the tangent at $\left(x,x^3\right)$ is $3x^2$.

Next up the ladder is $v(x)=x^4$. You have probably guessed that the slope of the tangent at $A\left(x,x^4\right)$ is $4x^3$. If YES, you guessed the right answer. To confirm it take a point $P(x+h,(x+h)^4)$ near A. The slope of the secant line joining A and P is

$$\begin{aligned}\frac{v(x+h)-v(x)}{h}&=\frac{(x+h)^4-x^4}{h}=\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4-x^4}{h}\\&=\frac{h\left(4x^3+6x^2h+4xh^2+h^3\right)}{h}=4x^3+h(6x^2+4xh+h^2)\end{aligned}$$

The slope of the tangent is the limit of $4x^3+h(6x^2+4xh+h^2)$ as h approaches 0, which is $4x^3$

The pattern points to the next conclusion, that the slope of the tangent to the graph of $w(x)=x^5$ at $\left(x,x^5\right)$ is $5x^4$.

In general, if n is a positive integer then the slope of the tangent to the graph of $f(x)=x^n$ at an arbitrary point (x,x^n) is nx^{n-1}.

As the next example shows, this formula also holds for negative integers.

Example 2 *The slope of the tangent to the graph of* $g(x) = x^{-1} = \frac{1}{x}$, $x \neq 0$ *at an arbitrary point* $\left(x, \frac{1}{x}\right)$ *is* $(-1)\, x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$. *For,*

$$\frac{g(x+h) - g(x)}{h} = \frac{\frac{1}{(x+h)} - \frac{1}{x}}{h} = \frac{x - (x+h)}{hx\,(x+h)} = \frac{-h}{hx\,(x+h)} = \frac{-1}{x\,(x+h)}.$$

The required slope is the limit of $\frac{-1}{x\,(x+h)}$ *as* h *approaches 0, which is*

$$\frac{-1}{x\,(x+0)} = -\frac{1}{x^2} = -x^{-2}.$$

The next example shows that the formula also holds for rational exponents.

Example 3 *Let* $h(x) = x^{\frac{1}{2}} = \sqrt{x}, x > 0$. *Then the slope of the tangent to the graph of at* $(x, \sqrt{x})$ *is* $\frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$. *We verify this by calculating* $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$. *Rationalizing the numerator gives:*

$$\begin{aligned} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\ &= \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)} \end{aligned}$$

Therefore $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{2}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$

In general, if r **is any real number and both** x^r **and** x^{r-1} **are defined then the slope of the tangent to the graph of** $f(x) = x^r$ **at** (x, x^r) **is** rx^{r-1}**. This is called the power rule for slopes of tangents.** You will prove it in the exercises when r is any rational number, positive or negative. The proof when r is not rational requires some "heavy duty tools" we do not develop in this course, therefore we will not give it.

Exercise 4

1. *Use the power rule to determine the slope of the tangent to the graph of the given function at the stated point:*

   (a) $f(x) = \frac{1}{x}$ *at* (i) $(1, 1)$, (ii) $\left(3, \frac{1}{3}\right)$, (iii) $\left(-2, -\frac{1}{2}\right)$

   (b) $f(x) = \frac{1}{\sqrt{x}}$ *at* (i) $(1, 1)$, (ii) $\left(4, \frac{1}{2}\right)$, (iii) $\left(9, \frac{1}{3}\right)$

   (c) $f(x) = x^{3/2}$ *at* (i) $(4, 8)$, (ii) $\left(2, 2^{3/2}\right)$, (iii) $\left(a, a^{3/2}\right)$

   (d) $f(x) = \frac{-1}{x^{5/3}}$ *at at* (i) $(1, -1)$, (ii) $\left(-8, \frac{1}{32}\right)$, (iii) $\left(a^3, -\frac{1}{a^5}\right)$

2. *There are two points on the graph of* $f(x) = x^5$ *where the slope of the tangent is 80. What are they?*

3. *What is the equation of the tangent to the graph of* $f(x) = x^4$ *at the point* $(-1, 1)$*?*

4. *Determine the equations of the two tangents to the graph of* $f(x) = 3x^{1/3}$ *which have slope* $\frac{1}{4}$.