# Inverse Hyperbolic Functions

Consider $f(x) = \sinh x$. Its graph reveals that it is a one-to-one function, therefore it has an inverse, which we denote by $\sinh^{-1} x$. To obtain a formula for $\sinh^{-1} x$, we write the equation

$$y = \sinh x \tag{1}$$

and solve for $x$ in terms of $y$. Clearly, (1) implies that

$$y = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x}$$

This may be re-arranged as $(e^x)^2 - 2ye^x - 1 = 0$, which is a quadratic equation in $e^x$. By the quadratic formula

$$e^x = \tfrac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$$

Since $e^x$ cannot be negative, and $\sqrt{y^2+1} > y$, we must choose the positive sign. Therefore $e^x = y + \sqrt{y^2+1}$. This implies that $x = \ln\left(y + \sqrt{y^2+1}\right)$. Therefore

$$\sinh^{-1} y = \ln\left(y + \sqrt{y^2+1}\right) \tag{2}$$

Now consider $g(x) = \cosh x$, for $x \geq 0$, (we have to restrict the values of $x$ to get a one-to-one function). Its inverse is denoted by $\cosh^{-1} x$. To determine its formula, we write the equation

$$y = \cosh x$$

then solve for $x$ in terms of $y$. Go through the same routine as above. You should obtain

$$e^x = y \pm \sqrt{y^2 - 1}$$

Since $e^x \to \infty$ as $y \to \infty$ while $y - \sqrt{y^2-1} \to 0$ as $y \to \infty$, we must choose the positive sign. Therefore $e^x = y + \sqrt{y^2-1}$. This implies that $x = \ln\left(y + \sqrt{y^2-1}\right)$. Therefore

$$\cosh^{-1} y = \ln\left(y + \sqrt{y^2-1}\right), \quad y \geq 1 \tag{3}$$

Lastly, note that the graph of $h(x) = \tanh x$ indicates that $h$ is one-to-one, therefore it has an inverse, denoted by $\tanh^{-1} x$. Write $y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and re-arrange to get $e^{2x} = \left(\frac{1+y}{1-y}\right)$. Deduce that

$$\tanh^{-1} y = \tfrac{1}{2} \ln\left(\frac{1+y}{1-y}\right), \quad -1 < y < 1$$

**Exercise 1**

1. *Verify that if* $u(x) = \cosh^{-1} x$, $v(x) = \sinh^{-1} x$, *and* $w(x) = \tanh^{-1} x$ *then*

$$u'(x) = \frac{1}{\sqrt{x^2-1}}, \quad v'(x) = \frac{1}{\sqrt{x^2+1}}, \quad w'(x) = \frac{1}{1-x^2}$$

2. *We showed above that if* $y = \cosh x$ *and* $y \geq 0$ *then* $e^x = y + \sqrt{y^2-1}$. *This implies that*

$$e^{-x} = \frac{1}{e^x} = \frac{1}{y+\sqrt{y^2-1}} = \frac{y-\sqrt{y^2-1}}{\left(y+\sqrt{y^2-1}\right)\left(y-\sqrt{y^2-1}\right)} = y - \sqrt{y^2-1}$$

*Since* $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$, *it follows that* $\sinh x = \sqrt{y^2-1}$. *Show in a similar way that if* $y = \sinh x$ *then* $\cosh x = \sqrt{y^2+1}$.

*These results useful in integration by substitution.*

$$\left.\begin{array}{l} \text{If } \cosh x = y \text{ then } \sinh x = \sqrt{y^2 - 1}. \\ \text{If } \sinh x = y \text{ then } \cosh x = \sqrt{y^2 + 1}. \end{array}\right\} \tag{4}$$

3. *The function* $f(x) = \sec x$ *is one-to-one on* $\left[0, \frac{1}{2}\pi\right]$ *therefore we may define an inverse function* $g(x) = \operatorname{arcsec} x$.

   *Show that its derivative is* $g'(x) = \dfrac{1}{x\sqrt{x^2 - 1}}$, $x > 1$

   *In an earlier exercise, you showed that the derivative of* $\arccos\left(\frac{1}{x}\right)$ *is also* $\dfrac{1}{x\sqrt{x^2-1}}$. *Is* $\operatorname{arcsec} x$ *related to* $\arccos\left(\frac{1}{x}\right)$ *or this is pure chance? Defend your answer.*

4. *The function* $f(x) = \csc x$ *is one-to-one on* $\left[0, \frac{1}{2}\pi\right]$ *therefore we may define an inverse function* $h(x) = \operatorname{arccsc} x$. *Determine its derivative.*

5. *Let* $f(x) = \ln\left(x + \sqrt{a^2 + x^2}\right)$ *where* $a$ *is a constant. Show that* $f'(x) = \dfrac{1}{\sqrt{a^2 + x^2}}$.