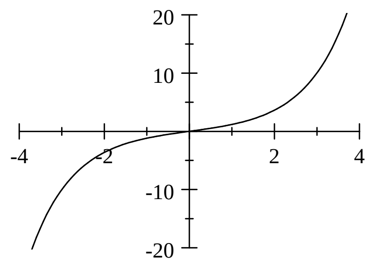
The Hyperbolic Functions

We define the hyperbolic sine, cosine, tangent and their inverses in this lecture and ask you to determine some of their properties in the exercises.

The hyperbolic sine function

The hyperbolic sine function is denoted by $\sinh x$, (pronounced "sine hyperbolic x", or "shine x" or "sinch x"). Its domain is the set of all real numbers and its formula is $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$. A section of its graph is given below.

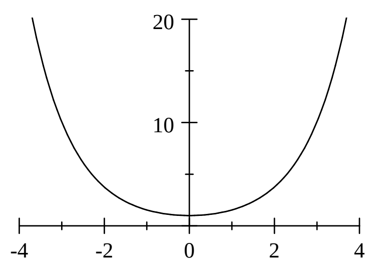


Unlike the trigonometric sine function, $\sinh x$ is not periodic and its values do exceed 1 and go below -1. It follows from the formula that:

1. $\sinh 0 = 0$.
2. $\sinh x \to \infty$ as $x \to \infty$ and $\sinh x \to -\infty$ as $x \to -\infty$.
3. $\sinh x$ has no critical point because its derivative $\frac{1}{2}\left(e^x + e^{-x}\right)$ is positive for all real numbers x.

The hyperbolic cosine function

The hyperbolic cosine function is denoted by $\cosh x$, (pronounced "cosine hyperbolic x", or "kosh x"). Its domain is the set of all real numbers and its formula is $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$. A section of its graph is given below.



Like the hyperbolic sine function, it is not periodic and its values do exceed 1. Its formula implies that:

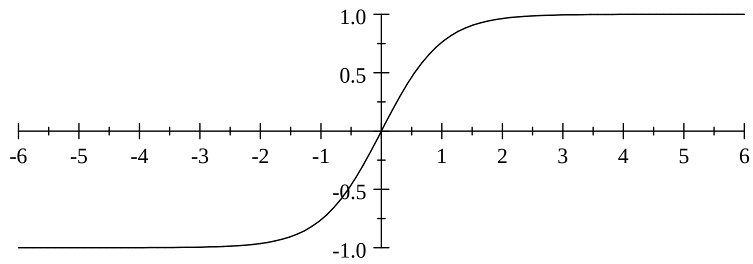
1. $\cosh 0 = 1$.
2. $\cosh x \to \infty$ as $x \to \infty$ and $\cosh x \to \infty$ as $x \to -\infty$.
3. It has a critical point $x = 0$, which is a point of relative minimum.

The hyperbolic tangent function

The hyperbolic tangent function is denoted by $\tanh x$, (pronounced "tan hyperbolic x"). Its domain is the set of all real numbers and its formula is

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

A section of its graph is given below



It has no critical points and all its values are between -1 and 1. This is because $e^x + e^{-x}$ is bigger than $|e^x - e^{-x}|$ for all real numbers x.

Three more hyperbolic functions

The hyperbolic secant, cosecant and cotangent are defined the same way the corresponding trigonometric functions are defined. Thus

$$\coth x = \frac{1}{\tanh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}, \quad \text{and} \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$

These functions satisfy identities that should remind you of some trigonometric identities. Examples:

1. Since $\cosh x + \sinh x = e^x$ and $\cosh x - \sinh x = e^{-x}$, it follows that

$$1 = e^x e^{-x} = (\cosh x + \sinh x)(\cosh x - \sinh x) = \cosh^2 x - \sinh^2 x$$

   The corresponding trigonometric identity is $1 = \cos^2 x + \sin^2 x$.

2. Since $\cosh^2 x - \sinh^2 x = 1$, dividing both sides of the identity by $\cosh^2 x$ gives $1 - \tanh^2 x = \operatorname{sech}^2 x$. The corresponding trigonometric identity is $\sec^2 x = 1 + \tan^2 x$.

3. Since $\cosh^2 x - \sinh^2 x = 1$, dividing both sides of the identity by $\sinh^2 x$ gives $\coth^2 x - 1 = \operatorname{csch}^2 x$. The corresponding trigonometric identity is $\csc^2 x = 1 + \cot^2 x$.

4. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$. To prove this, note that

$$\sinh x \cosh y + \cosh x \sinh y = \tfrac{1}{2}\left(e^x - e^{-x}\right)\tfrac{1}{2}\left(e^y + e^{-y}\right) + \tfrac{1}{2}\left(e^x + e^{-x}\right)\tfrac{1}{2}\left(e^y - e^{-y}\right)$$

   Expand and collect like terms. The result should be

$$\begin{aligned}\sinh x \cosh y + \cosh x \sinh y &= \tfrac{1}{4}\left(e^{x+y} + e^{x-y} - e^{y-x} - e^{-x-y}\right) + \tfrac{1}{4}\left(e^{x+y} - e^{x-y} + e^{y-x} - e^{-x-y}\right) \\ &= \tfrac{1}{2}\left(e^{x+y} - e^{-x-y}\right) = \sinh(x+y)\end{aligned}$$

   The corresponding trigonometric identity is $\sin(x + y) = \sin x \cos y + \cos x \sin y$

Exercise 1

1. *Show that the derivative of* $\sinh x$ *is* $\cosh x$.
2. *Show that the derivative of* $f(x) = \cosh x$ *is* $f'(x) = \sinh x$, *(NOT* $-\cosh x$*).*

3. *Use the quotient rule to show that the derivative of* $g(x) = \tanh x$ *is* $g'(x) = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = sech^2\, x$ *and that of* $h(x) = \coth x$ *is* $h'(x) = -csch^2\, x$.

4. *Show that the derivative of* $u(x) = sech\ x$ *is* $u'(x) = -sech\ x\ tanh\ x$ *and the derivative of* $v(x) = csch\ x$ *is* $v'(x) = -csch\ x\ coth\ x$

5. *Prove the following identities:*

(a) $\sinh(-x) = -\sinh x$	*(b)* $\cosh(-x) = \cosh x$
(c) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$	*(d)* $\cosh 2x = 2\sinh^2 x + 1$
(e) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$	*(f)* $\cosh 2x = 2\cosh^2 x - 1$
(g) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$	*(h)* $\sinh 2x = 2\sinh x \cosh x$

We record (d), (f) and (h) for later use in the following form:

$$\begin{array}{ll} i)\ \sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1) & ii)\ \cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1) \\ iii)\ \sinh x \cosh x = \tfrac{1}{2}\sinh 2x. & \end{array} \qquad (1)$$

6. *Show that:* *(a)* $\cosh(\ln x) = \frac{1}{2}\left(x + \frac{1}{x}\right)$, *(b)* $\sinh(\ln x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$, *(c)* $\tanh(\ln x) = \frac{x^2-1}{x^2+1}$.

7. *Use the chain rule to find the derivative of each function. Where they appear,* a *and* b *are constants.*

(a) $f(x) = 4\sinh 3x$	*(b)* $g(x) = a\cosh bx$	*(c)* $h(x) = \sqrt{3 + \cosh 2x}$
(d) $v(x) = 5\cosh\left(x^2\right)$	*(e)* $z(x) = \ln(\sinh ax)$	*(f)* $w(x) = 4\tanh\sqrt{x}$

8. *Determine the most general antiderivative of each function:*

a) $f(x) = 3\sinh 4x + 4\cosh 3x$	*b)* $g(x) = sech^2 2x - 3$
c) $h(x) = \sin 2x + \sinh 3x + 7$	*d)* $u(x) = 5e^{2x} + csch^2 \frac{1}{2}x$