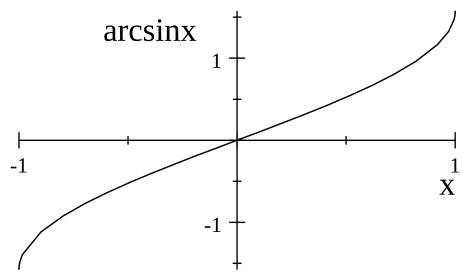
Derivatives of Inverse Functions

Implicit differentiation enables us to determine the derivatives of inverse functions. In this lecture, we determine the derivatives of $\arcsin x$, $\arccos x$, $\arctan x$, and $\ln x$.

To find the derivative of $\arcsin x$

Let $f(x) = \sin x$, $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$. Its inverse is $f^{-1}(x) = \arcsin x$, also written as $\sin^{-1}(x)$, (which you should not mistake for $1/\sin x$). Its graph is shown below.



Graph of $\arcsin x$

To simplify notation, write $f^{-1}(x) = \arcsin x$ as $y = \arcsin x$, (which one may read as "y is the angle whose sine is x"). Thus

$$\sin y = x \tag{1}$$

Differentiate both sides of (1) implicitly and solve for the derivative of y. You should get

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

But we need a derivative expressed in terms of x. To get it, we turn to the identity

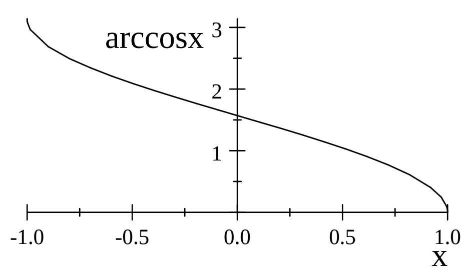
$$\cos^2 y + \sin^2 y = 1$$

It implies that $\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - x^2}$. Since the slope of the tangent at any point on the graph of y is positive, (see its graph), we must take the positive sign. Therefore if $y = \arcsin x$ then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

To find the derivative of $\arccos x$

Let $g(x) = \cos x$, $0 \leq x \leq \pi$. Its inverse is $g^{-1}(x) = \arccos x$, with graph shown below.



Graph of $\arccos x$

Once again, write $g^{-1}(x) = \arccos x$ as $y = \arccos x$, (to be read as "y is the angle whose cosine is x"). Thus

$$\cos y = x$$

When we differentiate implicitly and re-arrange the resulting equation we get

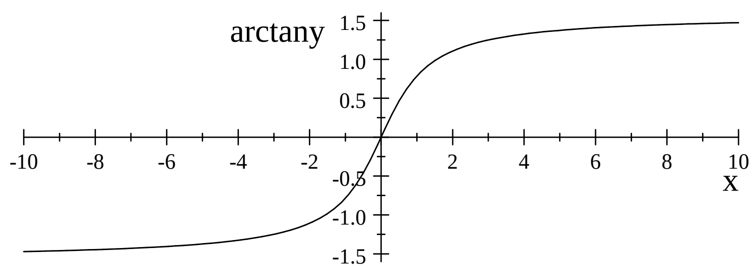
$$\frac{dy}{dx} = -\frac{1}{\sin y} = \pm\sqrt{\frac{1}{1-\cos^2 x}} = \pm\frac{1}{\sqrt{1-x^2}}$$

Since the slope of the tangent at any point on the graph of $\arccos x$ is negative, we must take the negative sign. Therefore if $y = \arccos x$ then

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

The derivative of $\arctan x$

Let $h(x) = \tan x$, $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$. Its inverse is $h^{-1}(x) = \arctan x$ with graph shown below.



Graph of $\arctan x$

Write $h^{-1}(x) = \arctan x$ as $y = \arctan x$, to be read "y is the angle whose tangent is x". It follows that $\tan y = x$. Take derivatives implicitly to get

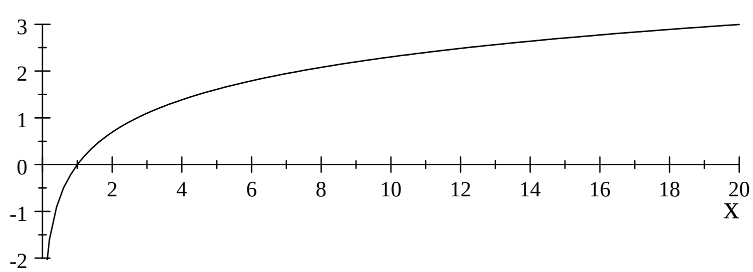
$$\left(\sec^2 y\right)\frac{dy}{dx} = \left(1+\tan^2 y\right)\frac{dy}{dx} = 1$$

We have used the identity $\sec^2 y = 1 + \tan^2 y$. Since $1 + \tan^2 y = 1 + x^2$, it follows that

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

To find the derivative of $\ln x$

If we let $w(x) = e^x$, where x is any real number, then its inverse $w^{-1}(x)$, $x > 0$, is called the natural logarithm function, and it is denoted by $\ln x$. More precisely, if $w(x) = e^x$ then $w^{-1}(x) = \ln x$. Its graph is given below.



Graph of $\ln x$

As before write $w^{-1}(x) = \ln x$ as $y = \ln x$. Then $e^y = x$. Differentiate implicitly and solve for the derivative. Since $e^y = x$, the result should be

$$\frac{dy}{dx} = \frac{1}{x} \tag{2}$$

As pointed out above, $\ln x$ is defined for positive values of x. For $x < 0$, we have to consider another function, namely $g(x) = \ln(-x)$. By the chain rule, its derivative is

$$g'(x) = \tfrac{1}{(-x)} \cdot (-1) = \tfrac{1}{x}$$

Thus the derivative of $\ln(-x)$ is also $\frac{1}{x}$. The two results may be stated as follows:

$$\text{If } \ h(x) = \ln|x| \ \text{ then } \ h'(x) = \frac{1}{x}$$

We summarize all these results in a table:

Function	$\arcsin x$	$\arccos x$	$\arctan x$	$\ln x$
Derivative	$\frac{1}{\sqrt{1-x^2}}$	$-\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$	$\frac{1}{x}$

The chain rule extends to these functions as follows:

Function	Derivative
$\arcsin(\quad)$	$\frac{1}{\sqrt{1-(\quad)^2}} \times$ Derivative of what is in $(\quad)$
$\arccos(\quad)$	$-\frac{1}{\sqrt{1-(\quad)^2}} \times$ Derivative of what is in $(\quad)$
$\arctan(\quad)$	$\frac{1}{1+(\quad)^2} \times$ Derivative of what is in $(\quad)$
$\ln\lvert\quad\rvert$	$\frac{1}{\lvert\quad\rvert} \times$ Derivative of what is in $\lvert\quad\rvert$

Exercise 1

1. *Calculate the derivative of each function and simplify your answer as much as possible. Where they appear, a and b are constants.*

a) $f(x) = x^2 \ln x$ *b)* $g(x) = \arcsin bx$ *c)* $u(x) = (\ln x)\, e^x$

d) $h(x) = 4 \arccos \frac{x}{3}$ *e)* $w(x) = \frac{3}{4} \arcsin x$ *f)* $v(x) = 2 \arctan bx$

g) $f(x) = \frac{2\arcsin(6x)}{3}$ *h)* $g(x) = \ln(9x) - 16$ *i)* $f(x) = \frac{5 \arctan x}{3}$

j) $v(x) = \arctan\left(\frac{1}{x}\right)$ *k)* $f(x) = \arcsin\left(\frac{1}{x}\right)$ *l)* $u(x) = \arctan\left(x^2\right)$

m) $g(x) = \sqrt{\ln x}$ *n)* $h(x) = \ln(\sin x)$ *o)* $v(x) = \ln\left(1 + 4x^3\right)$

p) $w(x) = \ln(1 + bx)$ *q)* $u(x) = \frac{4}{5} \arcsin(\frac{x}{4})$ *r)* $z(x) = \ln(bx)$

s) $h(x) = 5 \arctan(8x)$ *t)* $f(x) = \arctan\left(\frac{ax}{b}\right)$ *u)* $w(x) = 2 \arcsin\left(\sqrt{x}\right)$

v) $f(x) = \arcsin(ax + b)$ *w)* $v(x) = \ln \frac{x(x+5)}{x+3}$ *x)* $z(x) = \arcsin\left(\frac{ax}{b}\right)$

2. *A lake polluted by coliform bacteria is treated with bacteria agents. Environmentalists estimate that t days after the treatment the number N of viable bacteria per milliliter will be given by*

$$N(t) = 10t - \ln\left(\tfrac{t}{10}\right) - 30.$$

   (a) *Find the number of viable bacteria per milliliter for $t = 3$, $t = 4.5$, and $t = 7.5$ days.*

   (b) *Find and interpret the derivative of N for $t = 5$ days and $t = 9$ days.*

3. *Show that:*

   (a) *If $f(x) = \arcsin(\cos x)$ then $f'(x) = -1$*

   (b) *If $g(x) = \cos(\arcsin x)$ then $g'(x) = -\dfrac{x}{\sqrt{1-x^2}}$*

   (c) *If $w(x) = \ln(\sec x + \tan x)$ then $w'(x) = \sec x$.*

   (d) *If $u(x) = \ln(\sec x)$ then $u'(x) = \tan x$*

   (e) *If $h(x) = \arccos\left(\frac{1}{x}\right)$ then $h'(x) = \dfrac{1}{x\sqrt{x^2-1}}$*

   (f) *If $f(x) = \arctan\left(\frac{b}{x}\right)$ where b is a constant then $f'(x) = -\dfrac{b}{b^2+x^2}$*

4. *The function $f(x) = \sec x$ is one-to-one on $(0, \frac{1}{2}\pi)$ therefore we may define an inverse function $g(x) = \operatorname{arcsec} x$.*

   (a) *Show that its derivative is $g'(x) = \dfrac{1}{x\sqrt{x^2-1}}$, $x > 1$*

   (b) *You showed above that the derivative of $\arccos\left(\frac{1}{x}\right)$ is also $\dfrac{1}{x\sqrt{x^2-1}}$. Is $\operatorname{arcsec} x$ related to $\arccos\left(\frac{1}{x}\right)$ or this is pure chance? Defend your answer.*

5. *The function $f(x) = \csc x$ is one-to-one on $(0, \frac{1}{2}\pi)$ therefore we may define an inverse function $h(x) = \operatorname{arccsc} x$. Determine its derivative.*

6. *Given $h(x) = \cot(\arcsin x)$, determine $h'(x)$ and simplify your answer.*

7. *Given $g(x) = \ln(\csc x)$, determine $g'(x)$ and simplify your answer.*

8. *To find the derivative of a complicated function like $f(x) = \ln\dfrac{(x^2+1)(3x-1)}{(x^4+5)}$, first expand the logarithm expression:*

$$\ln\frac{(x^2+1)(3x-1)}{(x^4+5)} = \ln(x^2+1) + \ln(3x-1) - \ln(x^4+5) \tag{3}$$

   The derivatives of the expressions in the right hand side of (3) are much easier to determine. We obtain

$$f'(x) = \frac{2x}{x^2+1} + \frac{3}{3x-1} - \frac{4x^3}{x^4+5}$$

   Determine the derivative of each function below in a similar way:

   (a) $\ln\left[x^3(1+\sin x)^4\right]$ (b) $\ln\dfrac{x^2}{(x+2)(3x-5)}$

   (c) $\ln\dfrac{(x^2+3)^3}{\sqrt{3x-5}}$ (d) $\ln\sqrt{\dfrac{x+4}{x^2+4}}$

9. *Let p be a positive number. Use L'Hopital's rule to show that $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x^p} = 0$.*

10. *Assume that you are required to determine* $\lim_{x\to\frac{\pi}{4}} (\tan x)^{1/\left(x-\frac{\pi}{4}\right)}$. *A direct substitution of* $\frac{\pi}{4}$ *does not work because the exponent becomes infinite when* $x = \frac{\pi}{4}$. *Let* $f(x) = (\tan x)^{1/\left(x-\frac{\pi}{4}\right)}$. *Show that*

$$\ln(f(x)) = \frac{\ln(\tan x)}{x - \frac{\pi}{4}}.$$

Now use L'Hopital's rule to verify that $\lim_{x\to\frac{\pi}{4}} \ln(f(x)) = 2$, *then deduce the value of* $\lim_{x\to\frac{\pi}{4}} (\tan x)^{1/\left(x-\frac{\pi}{4}\right)}$.

   (a) *Determine* $\lim_{x\to 1} \left(x^{\csc \pi x}\right)$ *in a similar way.*

11. *Let* $n \geq 1$ *be a constant and* $f(x) = x^n - \ln x$, $x \geq 1$. *By applying the Mean Value Theorem to* f *on the interval* $[1, x]$, *show that* $f(x) - f(1) > 0$ *for all* $x > 1$. *Use this to deduce that* $x^n > 1 + \ln x$ *for all* $x > 1$.

12. *Suppose you are required to determine* $\lim_{x\to 0^+} x \ln x$. *In this form, it does not fall into one of the two forms to which we applied L'Hopital's rule earlier on. However, if we write* $x \ln x$ *as* $\left(\frac{\ln x}{\frac{1}{x}}\right)$, *it falls into the second form. Use L'Hopital's rule to determine* $\lim_{x\to 0^+} \left(\frac{\ln x}{\frac{1}{x}}\right)$.

Now that we have the derivative of $\ln x$, we can determine derivatives of other functions involving exponents.

Example 2 *We know that the derivative of the exponential function* $u(x) = e^x$, *to base* e *is* $u'(x) = e^x$. *However, the derivatives of exponential functions to other bases are a little different. For example, consider* $f(x) = 2^x$. *Its derivative is not quite* 2^x. *To determine it, write* $y = 2^x$ *then take logarithms of both sides to base* e. *The result is*

$$\ln y = \ln(2^x) = x \ln 2$$

Now that x *is no longer an exponent, we can easily take derivatives implicitly, with respect to* x, *(remember that* $\ln 2$ *is a constant), to get*

$$\frac{1}{y}\frac{dy}{dx} = \ln 2.$$

Since $y = 2^x$, *solving for* $\frac{dy}{dx}$ *gives*

$$f'(x) = \frac{dy}{dx} = (\ln 2)\, y = (\ln 2)\, 2^x.$$

You should be able to show, in a similar way, that if b *is any positive number then the derivative of* $v(x) = b^x$ *is* $v'(x) = (\ln b)\, b^x$. *This fact is worth recording:*

The derivative of b^x **is** $(\ln b)\mathrm{b}^x$

Example 3 *To determine the derivative of* $g(x) = (3x)^{x^2}$, *we write* $g(x)$ *as* y *to get* $y = (3x)^{x^2}$ *then take logarithms of both sides to base* e. *We end up expressing* $\ln y$ *as a product of two familiar functions:*

$$\ln y = x^2 \ln 3x$$

Taking derivatives of both sides, with respect to x, *gives*

$$\frac{1}{y}\frac{dy}{dx} = 2x \ln 3x + x^2 \cdot \frac{1}{3x} \cdot 3 = 2x \ln 3x + x = x\,(2 \ln 3x + 1)\,.$$

It follows that $g'(x) = \frac{dy}{dx} = x\,(2\ln 3x + 1)\, y = x\,(2\ln 3x + 1)\,(3x)^{x^2}$.

Example 4 *Let b be a positive number that is not equal to 1. To determine the derivative of $h(x) = \log_b x$, we use the change of base formula to write h in terms of the natural logarithm as*

$$h(x) = \frac{\ln x}{\ln b}$$

It follows that $h'(x) = \frac{1}{\ln b} \cdot \frac{1}{x} = \frac{1}{(\ln b)\, x}$.

Exercise 5

1. Here is another method of determining the derivative of $f(x) = b^x$: Since e^x is the inverse of $\ln x$, $b = e^{\ln b}$. Therefore
$$f(x) = \left(e^{\ln b}\right)^x = e^{x \ln b}$$
Now use the chain rule to determine $f'(x)$.

2. Determine the derivative of each function:

a) $u(x) = 5^{2x}$	b) $v(x) = \log_{10}\left(3 + 2x^2\right)$	c) $w(x) = (1+x)^{1/x}$
d) $f(x) = x^{2x}$	e) $g(x) = \ln \frac{3x(x+2)}{(x^2+1)(x^2+5)}$	f) $z(x) = \left(1 + \frac{1}{x}\right)^x$
g) $g(x) = \left(\sqrt{x}\right)^{x^2}$	h) $h(x) = (x^x)^x$	i) $u(x) = (x)^{x^x}$

3. Use the chain rule to find the derivative of

a) $f(x) = \arctan \sqrt{x}$ b) $g(x) = \arcsin\left(x^2\right)$ c) $h(x) = \arccos\left(\frac{1}{x}\right)$

4. Let a be a constant. Show that the derivative of $w(x) = \ln\left(x + \sqrt{a^2 + x^2}\right)$ is $w'(x) = \dfrac{1}{\sqrt{x^2 + a^2}}$

5. Let a be a constant and $v(x) = \ln\left(\dfrac{a + \sqrt{a^2 + x^2}}{x}\right)$. Show that $v'(x) = -\dfrac{a}{x\sqrt{a^2 + x^2}}$.

6. The pH of a substance is defined by the formula
$$\text{pH} = -\log\left[H^+\right]$$
where $[H^+]$ is the concentration of hydrogen ions in the substance, measured in moles per liter, (which is abbreviated to moles/L). A certain chemical reaction causes the concentration of hydrogen ions in a substance to increase at the rate of 0.004 (moles/L)/sec. Find and interpret the rate of change in pH when $[H^+] = 0.15$ moles/L.

7. When oxyhemoglobin is reduced to hemoglobin, the hemoglobins are able to bind more H^+ ions in the blood thus regulating the acid level of blood. A formula to calculate a patient's blood pH level is
$$\text{pH} = 6.1 + \log\left(\frac{x}{y}\right),$$
where x is the base concentration and y is the acid concentration. During hyperventilation, oxyhemoglobin is not reduced to hemoglobin quickly. The alveolar carbon dioxide pressure decreases, blood base level decrease and blood acid level increases. This condition is known as respiratory alkalosis where the arterial blood pH level exceeds 7.5

   (a) Find the blood base to acid concentration ratio (x/y) for alkalosis.

   (b) During alkalosis if $\dfrac{dx}{dt} = 0.5$ (mEq/L)/sec and $-\dfrac{dy}{dt} = 0.3$ (mEq/L)/sec when $x = 28$ mEq/L and $y = 2.6$ mEq/L, what is the rate of change in pH?

(c) Interpret and explain scenario (b) above.

(d) How many moles per liter of hydrogen ions must be present for alkalosis to be diagnosed?

8. Acid rain is toxic to vegetation and aquatic life. In northeastern U.S. forest soils are naturally acidic and their surface waters are mildly alkaline (basic). Acid rain increases the amount of positive hydrogen ions in the soil. If rainfall over a lake causes the pH to decrease at the rate of $-\frac{d\text{pH}}{dt} = 0.8/\text{hr}$, find the rate of change in hydrogen ions in the lake.

9. Give the most general antiderivative of the given function:

a) $f(x) = \frac{3}{x}$ b) $g(x) = \frac{2}{\sqrt{1-x^2}}$ c) $h(x) = \frac{3}{2\left(\sqrt{1-x^2}\right)}$

d) $h(x) = \frac{5}{4\,(x^2+1)}$ e) $u(x) = \frac{5}{4x^2+1}$ f) $v(x) = \frac{4}{5\,(x+2)}$

g) $w(x) = \frac{2}{x^2} - \frac{4}{x}$ h) $f(x) = \frac{x}{\sqrt{1-x^2}}$ i) $g(x) = x + \frac{1}{\sqrt{1-x^2}}$

j) $h(x) = x - \frac{1}{x^2+1}$ k) $u(x) = \frac{5}{x} - \frac{1}{x^2+1}$ l) $v(x) = \frac{6}{x} - \frac{1}{\sqrt{1-x^2}}$