Implicit Differentiation

Before we differentiate functions implicitly, we need to introduce other notations for the higher derivatives of a given function f, and introduce the concept of a function defined implicitly. The first step is to denote $f(x)$ by y, (or any other convenient letter, but y is the the most popular). Then:

The first order derivative of $y = f(x)$ *may be denoted by* $\dfrac{dy}{dx}$.
The second order derivative of $y = f(x)$ *is denoted by* $\dfrac{d^2y}{dx^2}$.
In general, the nth order derivative of $y = f(x)$ *is denoted by* $\dfrac{d^ny}{dx^n}$.

According to the chain rule, the derivative of $[f(x)]^2$ is $2\,[f(x)]\,f'(x)$ or simply $2f(x)f'(x)$. If we write $f(x)$ as y then $[f(x)]^2$ may be written as y^2 and $2f(x)f'(x)$ as $2y\dfrac{dy}{dx}$. Therefore the derivative of y^2 is $2y\dfrac{dy}{dx}$. In general:

The derivative of y^n *is* $ny^{n-1}\dfrac{dy}{dx}$.
The derivative of $\cos y$ *is* $-\sin y\dfrac{dy}{dx}$.
The derivative of e^y *is* $e^y\dfrac{dy}{dx}$.
The derivative of $\sec y$ *is* $\sec y\tan y\dfrac{dy}{dx}$.
The derivative of $\cot y$ *is* $-\csc^2 y\dfrac{dy}{dx}$.
The derivative of $\sin y$ *is* $\cos y\dfrac{dy}{dx}$.   We hope the pattern is clear

Turning to functions defined implicitly, take a function like $f(x) = \sqrt[3]{5x^2+7}$. We say it is defined directly because, given any number x in its domain, we evaluate the value $f(x)$ of f at x by simply substituting x into the right hand side. But this same function may be defined by the equation

$$(f(x))^3 - 5x^2 = 7 \tag{1}$$

This time, when a number, e.g. -2, is given, we do not get the value $f(-2)$ of f at -2 directly from substituting -2 into (1). We have to solve yet another equation, (which is $(f(-2))^3 - 20 = 7$) to get it. We say that equation (1) defines f indirectly or *implicitly.*

The following are more examples of functions defined implicitly. For convenience, we have written $f(x)$ as y. Thus, in each case, y is defined implicitly.

a. $3xy = 4$	b. $\sin xy = \frac{1}{2}x$	c. $x^2 + y^2 = 9$
d. $\sqrt{xy} - 4y^2 = 12$	e. $\sqrt{x+y} - 4x^2 = y$	f. $\frac{x+3}{y} = 4x^2 + y^2$.

It may be possible to write a function defined implicitly in a direct form. For example, if y is defined implicitly by $3xy = 4$, we may solve for y to get $y = \dfrac{4}{3x}$. However, there are functions for which the implicit form may be hard to convert into a direct form. Try y defined implicitly by

$$\frac{x+3}{y} = 4x^2 + y^2$$

The following is a procedure, called implicit differentiation or differentiating implicitly, for determining the derivative of a function y that is defined implicitly:

Take the derivative of *each term* in the equation defining y, using the rules for derivatives. The result should be an equation with at least one term involving the derivative of y. Solve the equation for the derivative.

Example 1 *Let y be defined implicitly by $y^3 - 5x^2 = 7$. To find $\frac{dy}{dx}$ we take the derivatives of the terms y^3, $-5x^2$, and 7, one at a time, to get*

$$3y^2\frac{dy}{dx} - 10x = 0$$

We now solve for $\frac{dy}{dx}$ and the result is $\frac{dy}{dx} = \frac{10x}{3y^2}$.

Example 2 *Let y be defined implicitly by $\frac{x+3}{y} = 4x^2 + y^2$. To find $\frac{dy}{dx}$, we have to find the derivative of each term in the equation. Before we do so, it is a good idea to first clear the fractions. Therefore, multiply both sides by y to get*

$$x + 3 = 4x^2y + y^3.$$

(This saves us a trip to the quotient rule.) Now take the derivatives of the terms x, 3, $4x^2y$, and y^3 one at a time. Note that $4x^2y$ is a product of $\left(4x^2\right)$ and y, therefore determining its derivative requires the use of the product rule. Its derivative is $8xy + 4x^2\frac{dy}{dx}$. The result of taking the derivative of each term is

$$1 + 0 = 8xy + 4x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx}.$$

Re-arrange this to get $1 - 8xy = \left(4x^2 + 3y^2\right)\frac{dy}{dx}$, then divide by $4x^2 + 3y^2$ to get the derivative:

$$\frac{dy}{dx} = \frac{1 - 8xy}{4x^2 + 3y^2}$$

Example 3 *Let y be defined implicitly by $\sin xy = \frac{1}{2}x - y^2$. To find $\frac{dy}{dx}$, take the derivative of the terms $\sin xy$, $\frac{1}{2}x$, and y^2 one at a time. Note that the term $\sin xy$ has the form $\sin(\quad)$, therefore its derivative is*

$$\cos xy \times \textit{Derivative of } (xy) \ \textit{ which equals } \ (\cos xy)\left(y + x\frac{dy}{dx}\right)$$

The result of taking the derivative of each term is

$$(\cos xy)\left(y + x\frac{dy}{dx}\right) = \frac{1}{2} - 2y\frac{dy}{dx}$$

Re-arranging gives $(x\cos xy + 2y)\frac{dy}{dx} = \left(\frac{1}{2} - y\cos xy\right)$. We then solve for $\frac{dy}{dx}$ and the result is

$$\frac{dy}{dx} = \frac{1 - 2y\cos xy}{2x\cos xy + 4y}$$

To get higher order derivatives, take derivatives more than once using the fact that the derivative of $\frac{dy}{dx}$ is denoted by $\frac{d^2y}{dx^2}$, the derivative of $\frac{d^2y}{dx^2}$ is denoted by $\frac{d^3y}{dx^3}$, ...

Example 4 *Let y be defined implicitly by $x^2y - 3y^3 = -2x$, and assume that we are required to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(1,1)$ on the graph of y. Differentiating each term gives $2xy + x^2\frac{dy}{dx} - 9y^2\frac{dy}{dx} = -2$ which we may re-arrange as*

$$\left(x^2 - 9y^2\right)\frac{dy}{dx} = -2 - 2xy. \tag{2}$$

As before, we solve for $\frac{dy}{dx}$. The result is

$$\frac{dy}{dx} = \frac{-2 - 2xy}{x^2 - 9y^2} = \frac{2 + 2xy}{9y^2 - x^2} \tag{3}$$

The value of $\frac{dy}{dx}$ at $(1,1)$ is obtained by substituting $x = 1$ and $y = 1$ into (3), and it is $\frac{4}{8} = \frac{1}{2}$. To get the second derivative, you may take the derivatives of both sides of (3), but if you wish to avoid using the quotient rule, take the derivative of both sides of (2) to get

$$\left(2x - 18y\frac{dy}{dx}\right)\frac{dy}{dx} + \left(x^2 - 9y^2\right)\frac{d^2y}{dx^2} = -\left(2y + 2x\frac{dy}{dx}\right).$$

We have just shown that at $(1,1)$, $\frac{dy}{dx} = \frac{1}{2}$, therefore the value of $\frac{d^2y}{dx^2}$ at $(1,1)$ is given by

$$[2 - 18(\tfrac{1}{2})](\tfrac{1}{2}) + (-8)\frac{d^2y}{dx^2} = -[2 + 2(\tfrac{1}{2})]$$

Solve to get $\frac{d^2y}{dx^2} = -\frac{1}{16}$.

Example 5 *Let y be defined implicitly by $x^2 + xy + y^2 = 7$. The point $(1,-3)$ is on the graph of y, (because when we substitute $x = 1$ and $y = 3$ in the left hand side of the equation, we get 7). Suppose we have to find the equation of the tangent at $(1,3)$. We first differentiate implicitly and the result is*

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \tag{4}$$

We then substitute $x = 1$ and $y = -3$ in (4), and solve for the derivative, to get $\frac{dy}{dx} = -\frac{1}{5}$. Therefore the slope of the tangent at $(1,-3)$ is $-\frac{1}{5}$. Its equation must be given by

$$(y - (-3)) = -\tfrac{1}{5}(x - 1),$$

which simplifies to $y = -\frac{1}{5}x - \frac{14}{5}$.

Exercise 6

1. *Use implicit differentiation to find $\frac{dy}{dx}$ given that y is defined implicitly by:*

   (a) $xy + x^2y^2 = 5$. (b) $x^3 - xy + y^3 = 1$. (c) $x^{1/2} + y^{1/2} = 1$.

   (d) $x^2 = \dfrac{x - y^2}{x + y}$ (e) $x^2 + xy - y^2 = 1$. (f) $x \sin y + 2y = 0$

2. *Show that if $e^y - x = 0$ then $\frac{dy}{dx} = \frac{1}{e^y}$ and deduce that $\frac{dy}{dx} = \frac{1}{x}$.*

3. *Show that if $\tan y - x = 0$ then $\frac{dy}{dx} = \frac{1}{\sec^2 y}$ and deduce that $\frac{dy}{dx} = \frac{1}{1 + x^2}$.*

4. *In each question, verify that the given point is on the curve defined implicitly by the given equation then find the equation of the tangent at that point.*

a) $xy + x^2y^2 = 6,\ (1,2)$ *b)* $x^2y^2 = 4y,\ (2,1)$

c) $y^2 + 2xy - 4 = 0,\ (0,2)$ *d)* $y^2 - 3x^2y = \cos x,\ (0,1)$

e) $x^3y^3 = 9y,\ (1,3)$ *f)* $\cos y + y + x = 2,\ (1,0)$

g) $x^2 + y^2 - xy = 7,\ (2,-1)$ *h)* $xe^y - 3y = 1,\ (1,0)$

5. *The function y is defined implicitly by $x^2 + 2x + y^2 - 3y = 0$. Determine the points on its graph where it has horizontal tangents. Also determine the points where it has vertical tangents, (i.e. the points where the tangent has infinite slope).*

6. *Find the second derivative* $\dfrac{d^2y}{dx^2}$ *(or y'') at the given point, if $y = f(x)$ is defined implicitly by the given equation:* *(a)* $x^3 + y^3 = 16$ *at* $(2,2)$

(b) $xy + y^2 = 1$ *at* $(0,-1)$ *(c)* $e^{xy} + 2y - 3x = \sin y$ *at* $\left(\frac{1}{3}, 0\right)$

7. *Find the second derivative* $\dfrac{d^2y}{dx^2}$ *(or y'') given that y is defined implicitly by the equation:*

(a) $x^2 + y^2 = 4$ *(b)* $y^2x^2 + 3x - 4y = 5$ *(c)* $e^{xy} + 2y - 3x = \sin y$

8. *Show that the tangent to the curve $y^2 = x^3 - 6x + 4$ at the point $A(-1,3)$ intersects the curve at another point B and give the coordinates of B.*

9. *Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What are their equations?*

10. *A normal to a given curve at a point (a,b) on the curve is a line that is perpendicular to the tangent at (a,b). Find the equation of the normal to the parabola $y^2 - 4x = 0$ at the point $(1,2)$.*