# L'Hopital's Rule

If $f$ and $g$ are functions such that $f(a) = g(a) = 0$ then when we try to compute $\lim_{x \to a} \frac{f(x)}{g(x)}$ by substituting $x = a$ into the numerator and denominator, we may end up with the undefined expression $\frac{0}{0}$. In such a case, L'Hopital's rule may save the day. It states that if $f(a) = g(a) = 0$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

To prove it, take any number $x$ close to $a$. By the Generalized Mean Value Theorem, there is a number $y$ between $a$ and $x$ such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(y)}{g'(y)}$$

Since $f(a) = g(a) = 0$, it follows that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{y \to a} \frac{f'(y)}{g'(y)}$$

The variable $y$ does not change the value of the limit. Therefore we may write

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

**Example 1** *Consider* $h(x) = \frac{x^3 - 1}{x^{14} - 1}$. *Let* $f(x) = x^3 - 1$ *and* $g(x) = x^{14} - 1$. *Then* $f(1) = g(1) = 0$ *and*

$$\lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \left( \frac{3x^2}{14x^{13}} \right) = \frac{3}{14}.$$

*By L'Hopital's rule,* $\lim_{x \to 1} h(x) = \lim_{x \to 1} \frac{x^3 - 1}{x^{14} - 1} = \lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{f'(x)}{g'(x)} = \frac{3}{14}$

**Example 2** *Consider* $h(x) = \frac{\sin 2x^2}{x^2}$. *Let* $f(x) = \sin 2x^2$ *and* $g(x) = x^2$. *Then* $f(0) = g(0) = 0$ *and*

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\left(\cos 2x^2\right) \cdot 4x}{2x} = \lim_{x \to 0} 2 \cos 2x^2 = 2.$$

*By L'Hopital's rule,* $\lim_{x \to 0} h(x) = \lim_{x \to 0} \frac{\sin 2x^2}{x^2} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 2$

The rule extends to cases in which derivatives also vanish. For example, suppose $f$ and $g$ are such that $f(a) = f'(a) = 0$ and $g(a) = g'(a) = 0$. If $\lim_{x \to a} \frac{f''(x)}{g''(x)}$ exists then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}.$$

In general, if $f(a) = f'(a) = \cdots = f^{(n-1)} = 0$; $g(a) = g'(a) = \cdots = g^{(n-1)} = 0$, and $\lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

**Example 3** *Consider* $\lim_{x\to 0}\frac{1-\cos 2x}{3x^2}$. *Let* $f(x) = 1-\cos 2x$ *and* $g(x) = 3x^2$. *Then* $f(0) = f'(0) = 0$ *and* $g(0) = g'(0) = 0$. *However,*

$$\lim_{x\to 0}\frac{f^{''}(x)}{g^{''}(x)} = \lim_{x\to 0}\frac{4\cos 2x}{6} = \frac{2}{3}.$$

*By L'Hopital's rule,*

$$\lim_{x\to 0}\frac{1-\cos 2x}{3x^2} = \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f^{''}(x)}{g^{''}(x)} = \frac{2}{3}$$

**Exercise 4** *Calculate the following limits:*

$$(a)\lim_{x\to 0}\left(\frac{\sin 3x}{\tan 5x}\right) \qquad (b)\lim_{x\to 2}\left(\frac{x^2-4}{x^3-8}\right) \qquad (c)\lim_{x\to 0}\frac{\tan\left(3x^2\right)}{4x^2}$$

$$(d)\lim_{x\to 0}\frac{x-\sin x}{2x^2} \qquad (e)\lim_{x\to 0}\left(\frac{x-\sin x}{x-x\cos x}\right) \qquad (f)\lim_{x\to -1}\left(\frac{x^4-2x^2+1}{x^3+4x^2+5x+2}\right)$$

## Another version of L'Hopital's rule

Suppose $f$ and $g$ are functions such that $f(x)\to\infty$ (or $-\infty$) and $g(x)\to\infty$ (or $-\infty$) as $x$ approaches $a$, ($a$ could be finite or infinite.)

$$\text{If } \lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ exists then } \lim_{x\to a}\frac{f(x)}{g(x)} \text{ also exists and it is equal to } \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

The proof happens to be considerably harder, so we skip it. Here are examples where it is used.

**Example 5** *Let* $f(x) = 3x+5$ *and* $g(x) = 4-5x$. *Then* $f(x)\to\infty$ *and* $g(x)\to-\infty$ *as* $x\to\infty$. *Since* $\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty} -\frac{3}{5}$, *it follows that* $\lim_{x\to\infty}\frac{3x+5}{4-5x} = -\frac{3}{5}$.

**Example 6** *Consider* $\lim_{x\to\infty}\frac{3x}{e^x}$. *Let* $f(x) = 3x$ *and* $g(x) = e^x$. *Then* $f(x)\to\infty$ *and* $g(x)\to\infty$ *as* $x\to\infty$. *But* $\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{3}{e^x} = 0$, *therefore* $\lim_{x\to\infty}\frac{3x}{e^x} = 0$.

**Example 7** *Let* $f(x) = x^2+5x-1$ *and* $g(x) = 2x^2-3x+5$. *Then* $f(x)$, $g(x)$, $f'(x)$, *and* $g'(x)$ *all approach* $\infty$ *as* $x\to\infty$. *Since* $\lim_{x\to\infty}\frac{f''(x)}{g''(x)} = \lim_{x\to\infty}\frac{2}{4} = \frac{1}{2}$, *it follows that* $\lim_{x\to\infty}\frac{x^2+5x-1}{2x^2-3x+5} = \frac{1}{2}$.

**Exercise 8** *Determine the required limits. In part (c), n is a positive integer.*

$$(a)\ \lim_{x\to\infty}\left(\frac{107x}{x^2+1}\right) \qquad (b)\ \lim_{x\to\infty}\frac{x^2}{e^x} \qquad (c)\ \lim_{x\to\infty}\frac{x^n}{e^x}$$

**Remark 9** *Problems* **??** *and* **??** *on page* **??** *provide applications of L'Hopital's rule to two other forms of indeterminate limits.*