Systems of Linear Equations

The following is an example that leads to a system of linear equations:

Example 1 *A Chemistry experiment requires 10 liters of a 20% acid solution. There are two storage tanks; one contains a 15% acid solution and the other contains a 35% acid solution. How many liters of each type may be mixed to produce 10 liters of the required 20% acid solution?*

> **Solution**: *Say we must mix x liters of the 15% acid solution with y liters of the 35% acid solution. Then*
> $$x + y = 10$$
> *because the mixture must be 10 liters. We now note that 10 liters of the 20% acid solution contain 2 liters of acid, (i.e. $10 \times \frac{20}{100} = 2$ liters of acid). These 2 liters must come from the two solutions to be mixed . The x liters of the 15% acid solution contain $x \times \frac{15}{100} = 0.15x$ liters of acid and the y liters of the 35% acid solution contain $y \times \frac{35}{100} = 0.35y$ liters of acid. These two quantities must add up to 2 liters, and so*
> $$0.15x + 0.35y = 2$$
> *Therefore we must determine numbers x and y that satisfy the following two equations:*
> $$\begin{aligned} x &+& y &=& 10 \\ 0.15x &+& 0.35y &=& 2 \end{aligned}$$
> *This is an example of a system of linear equations. It is called a 2×2 system because it contains 2 equations which are $x + y = 10$ and $0.15x + 0.35y = 2$; and the equations involve two unknowns, namely, x and y. A solution to the system is a pair of numbers which satisfy both equations. We will soon show that $x = 7.5$ and $y = 2.5$ is the only solution in this case. It is easy to show that the pair satisfies both equations:*
> $$x + y = 7.5 + 2.5 = 10 \qquad and \qquad 0.15x + 0.35y = 0.15 \times 7.5 + 0.35 \times 2.5 = 2$$
> *Determining the pair or pairs that satisfy all the equations in a given system of equations is called solving the system.*

Solving a System of Equations by Elimination

The following examples illustrates the process:

Example 2 *To solve the system of equation by elimination*
$$\begin{aligned} x &-& 3y &=& 11 \\ 4x &+& 5y &=& -7 \end{aligned}$$

> **Solution**: *As the term "elimination" suggests, we use one of the equations to write one of the variables in terms of the other variable. In this case it is easy to eliminate x; we use the first equation, namely $x - 3y = 11$, to write x in terms of y as $x = 3y + 11$. When we substitute this into the second equation, (there is no point substituting in the first equation because all you get is the statement $11 = 11$, which is true but of little use), we obtain $4(3y + 11) + 5y = -7$, which simplifies to*
> $$17y = -51$$
> *Thus x is "eliminated" and we can solve for y in the usual way. The result is $y = -3$. Now that we know y, we may find x by substituting this value of y into any one of the two given equations above. Actually, the easiest one is the modified one, $x = 3y + 11$, which we used to eliminate x. It gives $x = 3(-3) + 11 = 2$. Therefore the solution of the system is $x = 2$ and $y = -3$. It is common to give it as a pair $(2, -3)$. It is understood that the first number in the pair is the value of x and the second one is the value of y.*

Example 3 *To solve the system of linear equations by elimination*

$$4x \quad + \quad 3y \quad = \quad 8$$

$$5x \quad + \quad 2y \quad = \quad 3$$

Solution: *Let us use the second equation, namely $5x+2y=3$, to eliminate y. The result is $y = \dfrac{3}{2} - \dfrac{5x}{2}$. We now substitute this into the first equation to get*

$$4x + 3\left(\frac{3}{2} - \frac{5x}{2}\right) = 8$$

The variable y is eliminated, therefore we solve for x in the usual way. Multiply both sides by 2 to get rid of fractions. The result is

$$8x + 9 - 15x = 16 \qquad or \qquad -7x = 7$$

Therefore $x = -1$. Since $y = \dfrac{3}{2} - \dfrac{5x}{2}$ and $x = -1$, it follows that $y = \dfrac{3}{2} + \dfrac{5}{2} = \dfrac{8}{2} = 4$. We may give the solution as $x = -1$ and $y = 4$ or simply as the pair $(-1, 4)$.

The choice to use the second equation to eliminate y was purely arbitrary. We could have chosen to eliminate x using the first equation, i.e. the equation $4x + 3y = 8$. We would get

$$x = \frac{1}{4}\left(8 - 3y\right) = 2 - \frac{3y}{4}$$

Substituting in the second equation would then give

$$5\left(2 - \frac{3}{4}y\right) + 2y = 3 \qquad or \qquad 10 - \frac{15y}{4} + 2y = 3$$

This simplifies to

$$-\frac{7y}{4} = -7$$

Therefore $y = 4$. Substituting into $x = 2 - \dfrac{3y}{4}$ gives $x = 2 - \dfrac{3 \times 4}{4} = 2 - 3 = -1$

Example 4 *To solve the system*

$$x \quad + \quad y \quad = \quad 10$$

$$0.15x \quad + \quad 0.35y \quad = \quad 2$$

Solution: *We use the first equation to eliminate x. The result is $x = 10 - y$. We now substitute this into the second equation. The result is*

$$0.15\left(10 - y\right) + 0.35y = 2 \qquad or \qquad 0.2y + 1.5 = 2$$

This implies that $0.2y = 0.5$, hence $y = 2.5$. Substituting into $x = 10 - y$ gives $x = 10 - 2.5 = 7.5$. The solution is $(7.5, 2.5)$ as we pointed out above.

Solving Systems of Linear Equations Using Elementary Row Operations

This method involves systematic manipulation of the coefficients for the unknown variables x, y (or whatever they may be) to eliminate as many variables as possible from each equation in the system. It greatly simplifies the elimination process in the case of systems of 3 or more linear equations in 3 or more unknowns. To introduce it, consider the system we solved in Example 2. We ignore the variables x and y and write it simply as

$$\begin{pmatrix} 1 & -3 & | & 11 \\ & & | & \\ 4 & 5 & | & -7 \end{pmatrix}$$

This rectangular array of numbers is called an **augmented matrix** for the system of equations.

$\begin{pmatrix} 1 & -3 & | & 11 \end{pmatrix}$ is called the first row of the augmented matrix. We may denote it by R_1.

Naturally, $\begin{pmatrix} 4 & 5 & | & -7 \end{pmatrix}$ is called the second row of the augmented matrix and may be denoted by R_2.

To eliminate x from the second equation, we multiply R_1 by 4 and subtract the result from R_2. This is our first example of an elementary row operation. It is written next to the row it affects as shown below

$$\begin{pmatrix} 1 & -3 & | & 11 \\ 4 & 5 & | & -7 \end{pmatrix} \quad R_2 - 4R_1 \quad \begin{pmatrix} 1 & -3 & | & 11 \\ 0 & 17 & | & -51 \end{pmatrix}$$

The second equation was changed into $17y = -52$ which does not contain x. We actually obtained this equation above. Note that the first row is left unchanged. We simply used it to eliminate y from the second equation. The next step is to divide the second row by 17 in order to get y. This is also an elementary row operation and it is written, next to the row it affects, as shown below.

$$\begin{pmatrix} 1 & -3 & | & 11 \\ 0 & 17 & | & -51 \end{pmatrix} \quad R_2 \div 17 \quad \begin{pmatrix} 1 & -3 & | & 11 \\ 0 & 1 & | & -3 \end{pmatrix}$$

In Example 2, we substituted $y = -3$ into $x = 3y + 11$ to get x. Here we aim for the same result but in a slightly different way: *we use the equation $y = -3$ to eliminate y from the first equation.* Thus we multiply R_2 by 3 and add the result to R_1. The operation is written as shown below next to the row it affects.

$$\begin{pmatrix} 1 & -3 & | & 11 \\ 0 & 1 & | & -3 \end{pmatrix} \quad R_1 + 3R_2 \quad \begin{pmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -3 \end{pmatrix}$$

The last matrix simply reads $x = 2$ and $y = -3$.

You perform an elementary row operation when you do any one of the following to an augmented matrix:

1. Add a multiple of one row to another row.

2. Multiply or divide a row by a non-zero number.

3. Interchange two rows

Given a system of linear equations to solve, form its augmented matrix then apply these operations successively until you get a system whose solutions are easy to read off.

Example 5 *To solve the system*

$$\begin{aligned} 4x & + & 3y & = & 8 \\ 5x & + & 2y & = & 3 \end{aligned}$$

using row operations.

    Solution: The augmented matrix for the given system of equations is

$$\begin{pmatrix} 4 & 3 & | & 8 \\ 5 & 2 & | & 3 \end{pmatrix}$$

We plan to use the first equation to eliminate x from the second equation. To this end, divide the first row by 4 in order to get a coefficient 1 for x in the first equation.

$$\begin{pmatrix} 4 & 3 & | & 8 \\ 5 & 2 & | & 3 \end{pmatrix} \quad R_1 \div 4 \quad \begin{pmatrix} 1 & \frac{3}{4} & | & 2 \\ 5 & 2 & | & 3 \end{pmatrix}$$

3

Now we can easily eliminate x from the second equation. The operation is $R_2 - 5R_1$

$$\begin{pmatrix} 1 & \frac{3}{4} & | & 2 \\ & & | & \\ 5 & 2 & | & 3 \end{pmatrix} \quad R_2 - 5R_1 \quad \begin{pmatrix} 1 & \frac{3}{4} & | & 2 \\ & & | & \\ 0 & -\frac{7}{4} & | & -7 \end{pmatrix}$$

The next step is to use the second equation to eliminate y from the first equation. Therefore we divide the second equation by $-\frac{7}{4}$ in order to get a coefficient 1 for y. The operation is $R_2 \div \left(-\frac{7}{4}\right)$.

$$\begin{pmatrix} 1 & \frac{3}{4} & | & 2 \\ & & | & \\ 0 & -\frac{7}{4} & | & -7 \end{pmatrix} \quad R_2 \div \left(-\frac{7}{4}\right) \quad \begin{pmatrix} 1 & \frac{3}{4} & | & 2 \\ & & | & \\ 0 & 1 & | & 4 \end{pmatrix}$$

Now to eliminate y from the first equation, we multiply the second row by $\frac{3}{4}$ then subtract the result from the first equation.

$$\begin{pmatrix} 1 & \frac{3}{4} & | & 2 \\ & & | & \\ 0 & 1 & | & 4 \end{pmatrix} \quad R_1 - \frac{3}{4}R_2 \quad \begin{pmatrix} 1 & 0 & | & -1 \\ & & | & \\ 0 & 1 & | & 4 \end{pmatrix}$$

We now have a system whose solution is easy to read off. The solution is $x = -1$ and $y = 4$

In practice we give the above steps as follows:

$$\begin{pmatrix} 4 & 3 & | & 8 \\ & & | & \\ 5 & 2 & | & 3 \end{pmatrix} \quad R_1 \div 4 \quad \begin{pmatrix} 1 & \frac{3}{4} & | & 2 \\ & & | & \\ 5 & 2 & | & 3 \end{pmatrix} \quad R_2 - 5R_1 \quad \begin{pmatrix} 1 & \frac{3}{4} & | & 2 \\ & & | & \\ 0 & -\frac{7}{4} & | & -7 \end{pmatrix} \quad R_2 \div \left(-\frac{7}{4}\right)$$

$$\begin{pmatrix} 1 & \frac{3}{4} & | & 2 \\ & & | & \\ 0 & 1 & | & 4 \end{pmatrix} \quad R_1 - \frac{3}{4}R_2 \quad \begin{pmatrix} 1 & 0 & | & -1 \\ & & | & \\ 0 & 1 & | & 4 \end{pmatrix}$$

Exercise 6

1. Solve each system of equations

$$\text{(a)} \quad \begin{aligned} 3x &- 2y &= 12 \\ 4x &+ 3y &= -1 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} 3x &+ 4y &= -1 \\ x &+ 6y &= 2 \end{aligned} \qquad \text{(c)} \quad \begin{aligned} 2x &+ 5y &= 1 \\ x &+ 6y &= 2 \end{aligned}$$

2. In most areas of the United States, gas stations offer three grades of gasoline, indicated by octane ratings on the pumps, which are 87, 89, 93. When a tanker comes and delivers gas, it brings only two grades of gasoline, the highest and the lowest, filling two large underground tanks. If one chooses to buy the middle grade, the pump's computer mixes the two grades from the underground tanks appropriately. How many gallons of 87-octane gas and 93-octane gas should be blended in order to fill a customer's 15 gallon car with 89-octane gas?

3. A gala was held to fund-raise for a noble cause. The gate collection was $9.00 for each adult and $5.00 per child. A total of 357 people attended the gala and the gate collection came to $2601. How may children and how many adults attended?

4. An instructor would like to set a test with 24 questions to be graded out of 200 points. There should be multiple choice questions worth 4 points each and non-multiple choice questions worth 12 points each. How many questions of each type must he set?

5. The sum of the digits of a two-digit number is 10. When the digits are reversed, the number is decreased by 18. Find the number.

4