Higher Order Derivatives

A higher order derivative is *the derivative of a derivative*. For example, let $f(x) = 3x^2 - \frac{2}{x}$. Its derivative is $f'(x) = 6x + \frac{2}{x^2}$. The derivative of $f'(x)$ is an example of a higher order derivative of f, called the second derivative of f, and denoted by $f''(x)$. Thus

$$f''(x) = 6 - \frac{4}{x^3}.$$

One may also take the derivative of $f''(x)$. The result is called the third order derivative of f and is denoted by $f'''(x)$. Therefore $f'''(x) = \dfrac{12}{x^4}$. Fourth and higher order derivatives of f are calculated in a similar way.

For another example, consider $g(x) = 4x^4 - 3x^2 + 5x - 3$. Its first derivative is

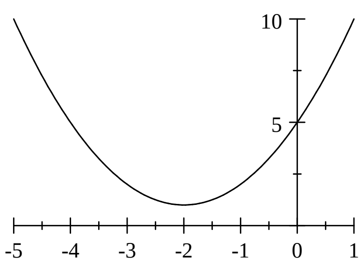
$$g'(x) = 16x^3 - 6x + 5.$$

The second derivative is $g''(x) = 48x^2 - 6$. The third derivative is $g'''(x) = 96x$, the fourth is written as $g^{(4)}(x)$ and it is the constant function $g^{(4)}(x) = 96$. The fifth and higher derivatives are all zeros.

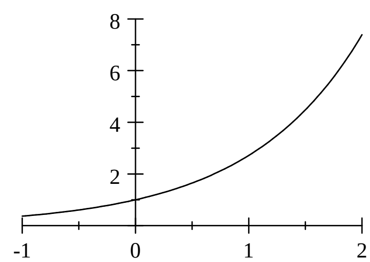
For practice, find the first and second derivative of each of the following:

(a) $f(x) = 3x^5 - 2x^3 + x - 3$	(b) $g(x) = 2\sin x - 4\cos x$	(c) $h(x) = 4e^{3x}$
(d) $u(x) = 8x\cos x - 1$	(e) $v(x) = 4 - x\sin x$	(f) $w(x) = \frac{2}{x^2}$
(g) $f(x) = \frac{4x}{7} - \frac{7}{4x^2} + 5$	(h) $g(x) = 5 - 3\sin\frac{1}{2}x$	(i) $h(x) = 2\sec \pi x$

Our first use of second order derivatives is to determine shapes of graphs. There are graphs like that of $f(x) = (x+2)^2 + 1$ and $g(x) = e^x$ shown below, shaped like right-side up bowls or part of right-side up bowls.

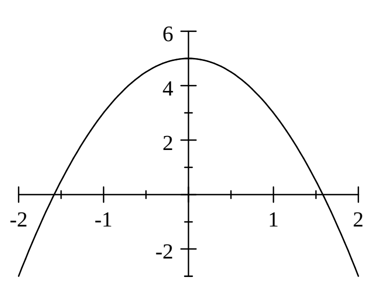


Graph of $f(x) = (x+2)^2 + 1$

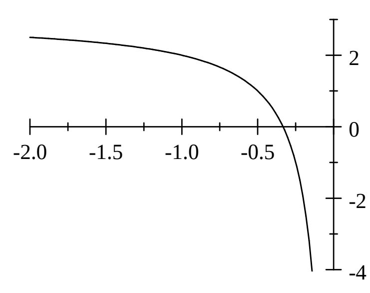


Graph of $g(x) = e^x$

We call them **concave up** graphs. A more precise definition is given in Exercise 12 on page 5. Their counter-parts, like the graphs of $h(x) = -2x^2 + 5$ and $v(x) = 3 + \frac{1}{x}$, $x < 0$, are shaped like upside-down bowls or part of upside-down bowls.

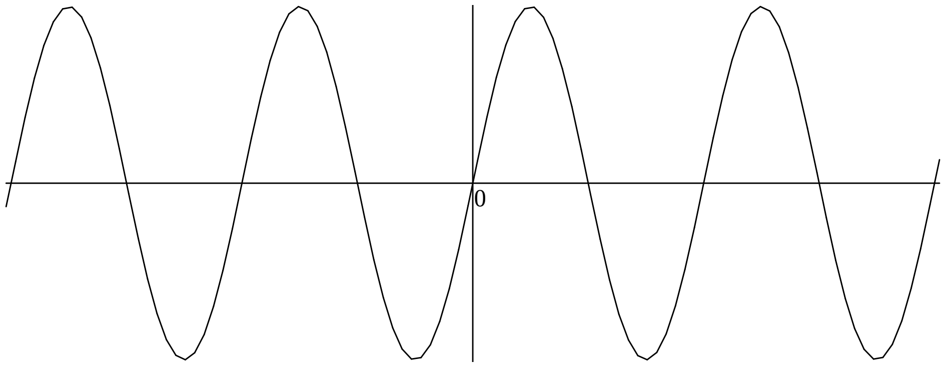


Graph of $h(x) = -2x^2 + 5$



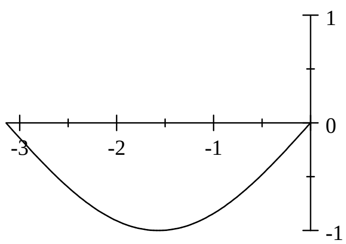
Graph of $v(x) = 3 + \frac{1}{x}$, $x < 0$

They are called **concave down** graphs. Some graphs are concave up on some intervals and concave down on others. An example is the graph of $\sin x$. It is concave down on intervals like $[0, \pi]$, $[2\pi, 3\pi]$, etc, and concave up on $[\pi, 2\pi]$ and many others.

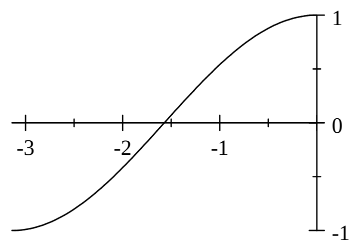


As the graphs of $f(x) = (x+2)^2 + 1$ and $g(x) = e^x$ demonstrate, if the graph of a function is concave up on an interval $[a, b]$ then the derivative of the function is increasing on the interval. (For the graph of $f(x) = (x+2)^2 + 1$ on the interval $[-5, 1]$, the $f'(x)$ is -6 at -5, it is zero at -2, it is 4 at 0, 5 at $\frac{1}{2}$, etc; so it is increasing. Actually, the derivative is $f'(x) - 2x + 4$, which is an increasing function.

Example 1 *The graph of $u(x) = \sin x$ is concave up on the interval $[-\pi, 0]$ and we have $u'(x) = \cos x$ which is increasing on $[-\pi, 0]$.*



Graph of u on $[-\pi, 0]$

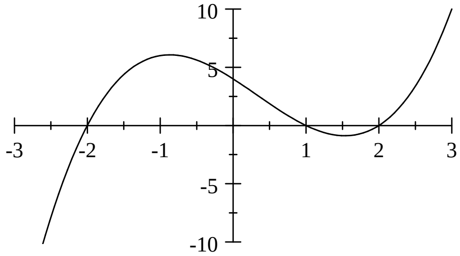


Graph of u' on $[-\pi, 0]$

Since an increasing function has a positive derivative, this suggests that if a given function has a concave up graph on an interval $[a, b]$ then its second derivative is positive on $[a, b]$, (because it is the derivative of the increasing function $f'(x)$). This is what we use here to identify concave up graphs. More precisely, *the graph of a function f is concave up on an interval $[a, b]$ if $f''(x)$, (the second derivative of f), is positive on $[a, b]$.*

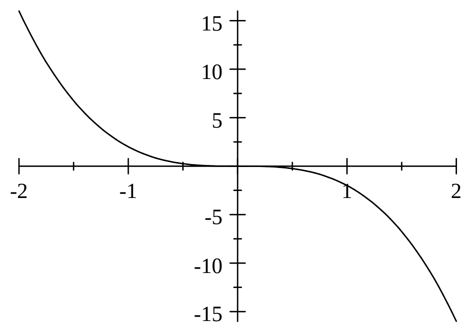
The graphs of $h(x) = -2x^2 + 5$ and $v(x) = 3 + \frac{1}{x}$, $x < 0$ suggest that if a function has a concave down graph on an interval $[a, b]$ then its derivative is decreasing, therefore its second derivative should be negative on $[a, b]$. We use this observation to identify concave down graphs. More precisely, *the graph of a function f is concave down on an interval $[a, b]$ if $f''(x)$, (the second derivative of f), is negative on $[a, b]$.*

Example 2 *Consider the function $f(x) = x^3 - x^2 - 4x + 4$. Its second derivative is $f''(x) = 6x - 2$. This is positive when $x > \frac{1}{3}$ and negative when $x < \frac{1}{3}$. Therefore its graph is concave up on the interval $\left(\frac{1}{3}, \infty\right)$, and concave down on $\left(-\infty, \frac{1}{3}\right)$.*



At the number $x = \frac{1}{3}$ the graph changes from being concave down, (to the left of $\frac{1}{3}$), to being concave up. Such a number is called a ***point of inflection*** *for the function.*

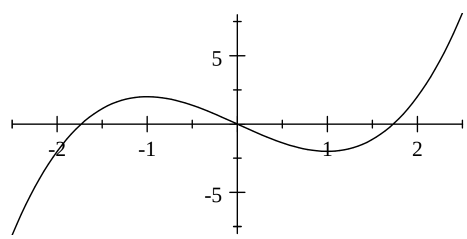
Example 3 *Let $g(x) = -2x^3$. Then $g''(x) = -12x$ which is positive on $(-\infty, 0)$ and negative on $(0, \infty)$. The graph is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.*



Its point of inflection is $x = 0$.

The Second Derivative Test for Maxima/Minima

If a number c is a point of relative maximum for a function f then the graph of f must be concave down on some interval $[a, b]$ containing c, (see the point $c = -1$ in the figure below). We observed that in such a case, the second derivative of f should be negative on the interval. In particular, $f''(c)$ should be negative.



The exact opposite happens when c is a point of relative minimum for f. In that case the graph of f must be concave up on some interval $[a, b]$ containing c, therefore $f''(x)$ should be positive on $[a, b]$. In particular, $f''(c)$ should be positive.

We already noted that a point of relative maximum or relative minimum for a function f is a critical point of f. Therefore the above observations lend support to the following **second derivative test** for the nature of a critical point:

- *If c is a critical point of a function f and $f''(c)$ is negative, then c is a point of relative maximum.*
- *If c is a critical point of a function f and $f''(c)$ is positive, then c is a point of relative minimum.*

For functions whose second derivatives are easy to determine, the second derivative test may be quicker to apply than the slope method we used in earlier

Example 4 *Consider the function $f(x) = 4 + 18x + \frac{3}{2}x^2 - 2x^3$ we met earlier, (page* **??***). Its critical points are the numbers x such that*

$$f'(x) = 18 + 3x - 6x^2 = -3(2x + 3)(x - 2) = 0$$

They are $x = 2$ and $x = -\frac{3}{2}$. To establish their nature, we determine the second derivative of f then evaluate it at 2 and $-\frac{3}{2}$. It turns out to be $f''(x) = 3 - 12x$ and

$$f''(2) = 3 - 24 = -21$$

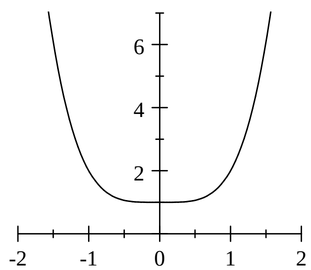
which is negative. Therefore $c = 2$ is a point of relative maximum for f. On the other hand,

$$f''(-\tfrac{3}{2}) = 3 - 12(-\tfrac{3}{2}) = 21$$

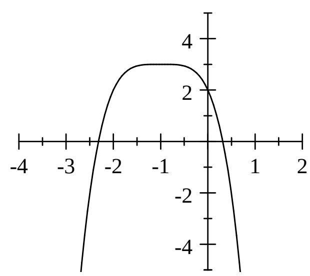
which is positive. Therefore $c = -\frac{3}{2}$ is a point of relative minimum for f.

Remark 5 *Unfortunately, if $f''(c) = 0$ at a critical point c, then the second derivative test shades no light on the nature of the critical point. As the example below show, it could be a point of relative minimum, a point of relative maximum, or neither.*

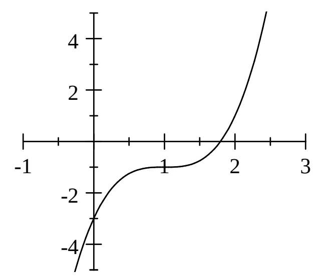
Example 6 *Let $f(x) = 1 + x^4$, $h(x) = 3 - (x+1)^4$, and $g(x) = 2(x-1)^3 - 1$. The second derivative of f ,(it happens to be $12x^2$), is zero at its critical point $c = 0$, which is a point of relative minimum. The second derivative of h ,(it happens to be $-12(x+1)^2$), is zero at its critical point $c = -1$, which is a point of relative maximum. The second derivative of g, (it happens to be $12(x-1)$), is zero at its critical point $c = 1$ which is neither a point of relative maximum nor a point of relative minimum.*



$f(x) = 1 + x^4$



$h(x) = 3 - (x+1)^4$



$g(x) = 2(x-1)^3 - 1$

> ***If you get a critical point c such that $f''(c) = 0$, use the slope method we introduced in Example ?? to establish its nature.***

Exercise 7

1. *Determine the interval or intervals where the graph of $w(x) = x^4 - x^2 + 3$ is concave up or concave down. What are the points of inflection of w (if any)?*

2. *Determine the interval or intervals where the graph of $h(x) = x^5 - x^3 + 5$ is concave up or concave down. What are the points of inflection of h (if any)?*

3. *In each case determine the critical point(s) of the given function then use the second derivative test to establish the nature of every critical point.*

   (a) $u(x) = x^2 + 3x + 1$ *(b)* $g(x) = x^4 - 4x^2 - 3$ *(c)* $f(x) = xe^x$

4. *For the function $f(x) = \sqrt{x^2 + 4x + 7}$, which of the two tests; the slope method of Example* **??** *on page* **??** *and then the second derivative test, is quicker to establish the nature of its critical point?*

5. *Use an appropriate method to establish the nature of each critical point of the given function, then sketch its graph.*

   (a) $h(x) = x^2(6 - x)$ *(b)* $w(x) = x^4 - 2x^2 + 3$ *(c)* $g(x) = x^6 - 3x^4$

   (e) $f(x) = \frac{x^2-1}{x-4}$ *(f)* $v(x) = x + 4/x$ *(g)* $u(x) = \frac{x}{9+x^2}$

6. *Show that if $f(x) = \frac{x^2}{\sqrt{x^2-4}}$ then $f'(x) = \frac{x^3-8x}{(x^2-4)^{3/2}}$. Now determine its vertical asymptotes and its critical points, (there are three of them). Establish the nature of each critical point then sketch the graph.*

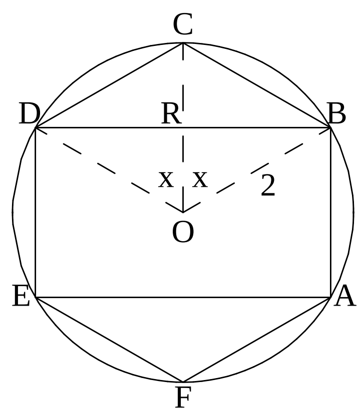
7. *You are given f with formula $f(x) = x^4 - 32x + 4$.*

   (a) *Determine its critical point, (it has only one), and establish its nature.*
   (b) *Sketch the graph of f and use it to estimate the roots of the equation $x^4 - 32x + 4 = 0$*
   (c) *Use Newton's method to determine the smallest root accurately to 3 decimal places. (Only the smallest root is required more accurately.)*

8. *Let n be a positive integer. Determine the nth derivative of:*

$$\text{(a) } f(x) = \sin x \quad \text{(b) } g(x) = e^{ax} \quad \text{(c) } u(x) = xe^x \quad \text{(d) } h(x) = \frac{1}{x}$$

9. *In the given figure, the circle has center O and radius 2 units. BD and OC intersect at R, $ABDE$ is a rectangle, BCD is an isosceles triangle and so is EFA, with $CB = CD = FA = FE$. Each of the angles BOC and DOC is x radians.*



   (a) *Show that RC has length $2 - 2\cos x$ and BD has length $4 \sin x$.*
   (b) *Prove that the area of the polygon $ABCDEF$ is $8 \sin x + 8 \sin x \cos x$.*
   (c) *What is the largest possible area of such a polygon?*

10. *According to the product rule, the derivative of $f(x)g(x)$ is $f'(x)g(x) + g'(x)f(x)$. Show that the second derivative of $f(x)g(x)$ is*

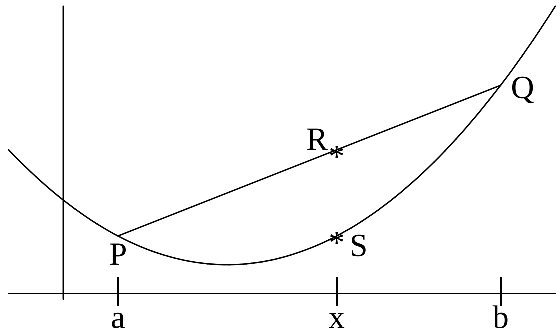
$$f''(x)g(x) + 2f'(x)g'(x) + g''(x)f(x)$$

   (a) *What is the third derivative of $f(x)g(x)$?*
   (b) *What is the nth derivative of $f(x)g(x)$, where n is a positive integer?*

11. *Sketch on the same axes, the graphs of $f(x) = \sin x$, (x in radians), and $g(x) = x - 1$. Use the sketches to determine an approximate root of the equation $\sin x = x - 1$ then use Newton's method to calculate it accurately to 4 decimal places.*

12. ***A more precise definition of a concave up graph**:*

   (a) *The graph of a function f is concave up on an interval if it satisfies the following condition: Given any two points $P(a, f(a))$ and $Q(b, f(b))$ on the graph, no point R on the chord PQ is below the graph of f. (See the figure below.)*



The equation of the chord PQ is $y(x) = \left[\frac{f(b)-f(a)}{b-a}\right](x-a)+f(a)$. *A number x between a and b may be written as*

$$x = (1-t)\,a + tb$$

where $0 \le t \le 1$. *If* $R(x, y(x))$ *is a point on the chord PQ and S is the point on the graph of f with the same x coordinate, show that R has coordinates* $((1-t)\,a+bt, (1-t)f(a)+tf(b))$ *and S has coordinates* $((1-t)\,a+bt, f((1-t)a+tb))$. *Use the fact that S must be below R in order for the graph to be concave up to deduce that*

$$f((1-t)a+tb) \le (1-t)f(a)+tf(b)$$

for all $0 \le t \le 1$. *Thus* ***the graph of a function*** f ***is concave up if for any points*** a ***and*** b ***in its domain and any number*** t ***between*** 0 ***and*** 1, $f((1-t)a+tb) \le (1-t)f(a)+tf(b)$.

(b) In this part of the question, you are required to show that if the second derivative of a function f is positive on an interval $[a, b]$ *then its graph is concave up; i.e.* $f((1-t)a+tb) \le (1-t)f(a)+tf(b)$ *for all points a and b in the interval and for all* $0 \le t \le 1$. *To get started, use the mean value theorem to show that there is a number* θ *between b and* $(1-t)a+tb$ *such that*

$$f(b) - f\left((1-t)a+tb\right) = (b-a)\,(1-t)\,f'(\theta) \qquad (1)$$

Multiply both sides of (1) by t and solve to get

$$tf(b) = tf\left((1-t)a+tb\right) + t\,(1-t)\,(b-a)\,f'(\theta) \qquad (2)$$

Also show that there is a number α *between a and* $(1-t)\,a+tb$ *such that*

$$(1-t)\,f(a) = -t\,(1-t)\,(b-a)\,f'(\alpha) + (1-t)\,f\left((1-t)\,a+tb\right) \qquad (3)$$

Add (2) to (3) and deduce that

$$(1-t)f(a)+tf(b) = f((1-t)a+tb) + t\,(1-t)\,(b-a)\,(f'(\theta)-f'(\alpha))$$

Use the Mean value Theorem to show that $(f'(\theta)-f'(\alpha)) \ge 0$ *then deduce that* $(1-t)f(a)+tf(b) \ge f((1-t)a+tb)$