Logarithm Functions

Logarithm functions are the inverses of exponential functions. To see the need for inverses of exponential functions, go back to the example of a car whose value decreases by $\frac{1}{8}$ every year. We deduced that its value x years after being purchased for \$18000 is $V(x) = 18000\left(\frac{7}{8}\right)^x$. How long does it take the value to drop below \$3500?

To answer the question, we must find an integer n such that $18000\left(\frac{7}{8}\right)^n < 3500$. One way of doing this is to find the number t such that $18000\left(\dfrac{7}{8}\right)^t = 3500$ then take the smallest integer bigger than t. When we divide both sides of this equation by 18000, we find that t must satisfy the equation

$$\left(\frac{7}{8}\right)^t = \frac{3500}{18000} = \frac{7}{36}$$

Therefore we must answer the question: **What exponent of $\dfrac{7}{8}$ equals $\dfrac{7}{36}$?** That exponent is called the logarithm of $\dfrac{7}{36}$ to base $\dfrac{7}{8}$. Therefore we must be capable of computing logarithms.

To introduce logarithms, consider the exponential function f with formula $f(x) = 2^x$. The following is a table of its sample values.

x	-2	-1	0	0.5	1	1.5	2	2.5	x
$f(x)$	0.25	0.5	1	$\sqrt{2}$	2	2.82	4	5.64	2^x

A table of sample values for its inverse is given below. It is obtained by simply swapping the rows for f. It is called the logarithm function to base 2 and it is denoted by $\log_2 x$.

x	0.25	0.5	1	$\sqrt{2}$	2	2.82	4	5.64	x
$f^{-1}(x)$	-2	-1	0	0.5	1	1.5	2	2.5	$\log_2 x$

Thus $\log_2 x$ is the exponent of 2 which equals x. For example, $\log_2 64 = 6$ because $2^6 = 64$, $\log_2 8 = 3$ because $2^3 = 8$, $\log_2 128 = 7$ because $2^7 = 128$, $\log_2\left(\frac{1}{8}\right) = -3$ because $2^{-3} = \frac{1}{8}$ and $\log_2\left(\frac{1}{16}\right) = -4$ because $2^{-4} = \frac{1}{16}$

In general, given a positive base b different from 1, the logarithm function to base b is the inverse of the exponential function $f(x) = b^x$. It is denoted by $\log_b x$. Clearly, if x is a positive number then $\log_b x$ is the exponent of b which equals x. Use your knowledge of exponents to complete the following table.

$\log_2 32 =$	$\log_3 9 =$	$\log_5 25 =$
$\log_2\left(\frac{1}{4}\right) =$	$\log_3\left(\frac{1}{81}\right) =$	$\log_5\left(\frac{1}{5}\right) =$
$\log_2 256 =$	$\log_3 243 =$	$\log_5 625 =$
$\log_2 2 =$	$\log_3 3 =$	$\log_5 5 =$
$\log_2 1 =$	$\log_3 1 =$	$\log_5 1 =$
$\log_2 4 =$	$\log_3 27 =$	$\log_5 125 =$
$\log_2\left(\frac{1}{64}\right) =$	$\log_3\left(\frac{1}{3}\right) =$	$\log_5\left(\frac{1}{25}\right) =$

The two most common logarithm functions are $\log_{10} x$, (the inverse of 10^x), and $\log_e x$ (the inverse of e^x). If you completed the tables for sample values of $f(x) = 10^x$ and $g(x) = e^x$ on page **??** you should have obtained the following tables:

x	1.3	4.22	3.05	0.6	-0.8	-1.7	-2.32	0.07	5.3
$f(x) = 10^x$	19.953	16596	1220.0	1.1482	0.15849	0.01995	0.00478	1.1749	199530

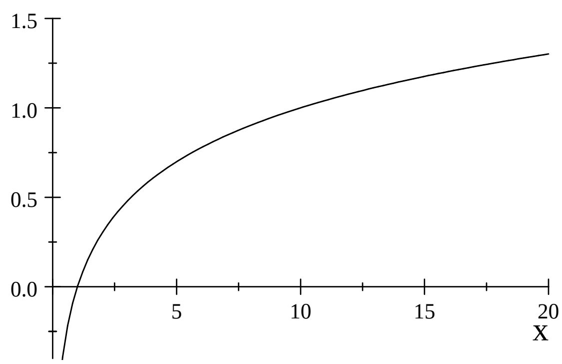
x	-1.7	3.55	4.17	0.8	-0.9	-1.68	-2.05	0.09	5.6
$g(x) = e^x$	0.18268	34.813	64.715	2.2255	0.40657	0.18637	0.12873	1.0942	270.43

Thus sample values for $\log_{10} x$ and $\log_e x$ are:

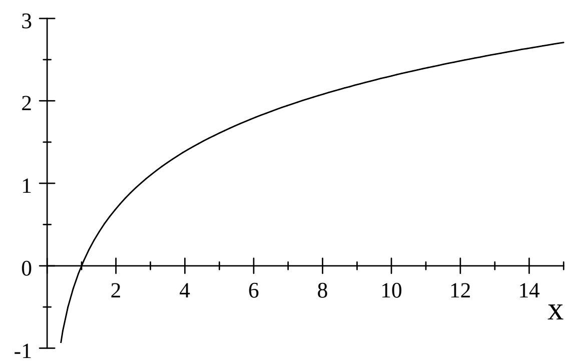
x	19.953	16596	1220.0	1.1482	0.15849	0.01995	0.00478	1.1749	199530
$\log_{10} x$	1.3	4.22	3.05	0.6	-0.8	-1.7	-2.32	0.07	5.3

x	0.18268	34.813	64.715	2.2255	0.40657	0.18637	0.12873	1.0942	270.43
$\log_e x$	-1.7	3.55	4.17	0.8	-0.9	-1.68	-2.05	0.09	5.6

The good news is that many calculators have these two functions. Their graphs are given below.



Graph of $f(x) = \log_{10} x$



Graph of $f(x) = \log_e x$

The symbols $\log_{10}$ and $\log_e x$ are abbreviated to $\log x$ and $\ln x$ respectively. For example, $\log 25$ is an abbreviation for $\log_{10} 25$ and $\ln 100$ is an abbreviation for $\log_e 100$.

Some Properties of Logarithms

Take any positive base $b \neq 1$. Let x and y be positive numbers. There is an exponent u such that $b^u = x$. In fact u is what we called the logarithm of x to base b and we write $u = \log_b x$. Likewise, there is an exponent v such that $b^v = y$ and we write $v = \log_b y$. Using rules of exponents yields

$$xy = b^u b^v = b^{u+v}$$

Therefore, the exponent of b that equals xy is $u + v$. In the language of logarithms,

$$\log_b xy = u + v = \log_b x + \log_b y$$

When we divide x by y and use rules of indices we get

$$\frac{x}{y} = \frac{b^u}{b^v} = b^{u-v}$$

Thus the exponent of b that equals $\frac{x}{y}$ is $u - v$. In the language of logarithms,

$$\log_b\left(\tfrac{x}{y}\right) = u - v = \log_b x - \log_b y$$

Finally, note that

$$\log_b x^2 = \log_b xx = \log_b x + \log_b x = 2\log_b x$$

$$\log_b x^3 = \log_b xxx = \log_b x + \log_b x + \log_b x = 3\log_b x$$

$$\log_b x^4 = \log_b xxxx = \log_b x + \log_b x + \log_b x + \log_b x = 4\log_b x$$

In general, if n is any real number, (not only whole numbers), then $\log_b x^n = n\log_b x$

We have derived three useful properties that are used in solving problems involving exponents and logarithms. We state them formally:

Let $b \neq 1$ be a positive number, x and y be any positive numbers, and n be any number. Then

$$\log_b xy = \log_b x + \log_b y \tag{1}$$

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \tag{2}$$

$$\log_b x^n = n\log_b x \tag{3}$$

Example 1 *Let b be a positive number such that $\log_b 5 = 4.33$ and $\log_b 3 = 2.96$ and $\log_b 2 = 1.87$ (rounded to 2 decimal places). Then:*

1. $\log_b 15 = \log_b 5 + \log_b 3 \simeq 4.33 + 2.96 = 7.29$ *(by property (1))*
2. $\log_b 0.6 = \log_b\left(\frac{3}{5}\right) = \log_b 3 - \log_b 5 \simeq 2.96 - 4.33 = -1.37$ *(by property (2))*
3. $\log_b 72 = \log_b(8 \times 9) = \log_b 8 + \log_b 9 = \log_b 2^3 + \log_b 3^2 = 3\log_b 2 + 2\log_b 3 \simeq 3(1.87) + 2(2.96) = 11.53$ *(by property (1) and (3))*
4. $$\begin{aligned}\log_b\sqrt{120} &= \log_b 120^{\frac{1}{2}} = \frac{1}{2}\log_b 120 = \frac{1}{2}\log_b(8\times 3\times 5)\\ &= \frac{1}{2}\left(\log_b 2^3 + \log_b 3 + \log_b 5\right) \simeq \frac{1}{2}\left(3(1.87) + 2.96 + 4.33\right) = 6.45\end{aligned}$$

Example 2 *To expand $\log_5\left(\frac{a^2b^4c}{125}\right)$ as much as possible:*

$$\begin{aligned}\log_5\left(\frac{a^2b^4c}{125}\right) &= \log_5\left(a^2b^4c\right) - \log_5 125 \text{ by property } (2)\\ &= \log_5 a^2 + \log_5 b^4 + \log_5 c - 3 \text{ by property } (1) \text{ and our knowledge that } \log_5 125 = 3\\ &= 2\log_5 a + 4\log_5 b + \log_5 c - 3 \text{ by property } (3)\end{aligned}$$

Example 3 *To condense each of (i)* $\log_3 x + \log_3 y - \log_3 2w$*, (ii)* $\log_3 a - \log_3 b - \log_3 4w$*, (iii)* $\log_2 3x^2 + \log_2 4y^3 - \log_2 6xy^2$*, (iv)* $2\log x - 3\log y + 4\log xy$ *into the logarithm of a single term:*

$$\begin{aligned} \log_3 x + \log_3 y - \log_3 2w &= \log_3 xy - \log_3 2w \text{ (using property (1) in reverse)} \\ &= \log_3 \frac{xy}{2w} \text{ (using property (2) in reverse).} \end{aligned}$$

$$\begin{aligned} \log_3 a - \log_3 b - \log_3 4w &= \log_3 a - (\log_3 b + \log_3 4w) \\ &= \log_3 a - \log_3 4bw \text{ (using property (1) in reverse)} \\ &= \log_3 \frac{a}{4bw} \text{ (using property (2) in reverse).} \end{aligned}$$

$$\begin{aligned} \log_2 3x^2 + \log_2 4y^3 - \log_2 6xy^2 &= \log_2 12x^2y^3 - \log_2 6xy^2 \text{ (using property (1) in reverse)} \\ &= \log_2 \frac{12x^2y^3}{6xy^2} = \log_2 2xy \text{ (using property (2) in reverse).} \end{aligned}$$

$$\begin{aligned} 2\log x - 3\log y + 4\log xy &= \log x^2 - \log y^3 + \log (xy)^4 \text{ (using property (3) in reverse)} \\ &= \log \frac{x^2}{y^3} + \log (xy)^4 \text{ (using property (2) in reverse)} \\ &= \log \frac{x^2 (xy)^4}{y^3} = \log x^6 y \text{ (using property (1) in reverse)} \end{aligned}$$

Example 4 *Consider the problem we posed at the beginning of this section, to determine how long it will take the value of the vehicle bought for $18000 to drop below $3500. We reduced it to determining a number t such that*

$$\left(\frac{7}{8}\right)^t = \frac{7}{36}$$

The complication here is that the variable is an exponent. To bring it down from up there, take logarithms of both sides. We may take logarithms to base 10 or e (because they are on common calculators). Say we take logarithms to base e. The result is

$$\ln\left(\frac{7}{8}\right)^t = \ln\frac{7}{36}$$

Property (3) of logarithms enables us to simplify this to

$$t\ln\left(\frac{7}{8}\right) = \ln\left(\frac{7}{36}\right) \tag{4}$$

Now t is no longer an exponent. Since $\ln\left(\frac{7}{8}\right) = -0.134$ *and* $\ln\left(\frac{7}{36}\right) = -1.638$*, Equation (4) may be written as*

$$-0.1335t = -1.638$$

which we may solve to get $t = (-1.638) \div (-0.1335) = 12.27$*. Therefore the value of the car will fall below $3500 after the 12th year.*

Example 5 *To solve for x given that* $\log_3 x + \log_3 (x-2) - \log_3 (x+2) = 1$*:*

Since we defined logarithms for positive numbers only, x must be bigger than 2. To solve the equation, we first condense the left hand side into the logarithm of a single term. The result is

$$\log_3 \frac{x(x-2)}{(x+2)} = 1$$

There is only one number whose logarithm to base 3 is 1 and that number is 3. Therefore $\frac{x(x-2)}{(x+2)} = 3$. *This implies that*

$$x(x-2) = 3(x+2) \quad \text{or} \quad x^2 - 5x - 6 = 0$$

Since $x^2 - 5x - 6$ factors as $(x-6)(x+1)$, the equation $x^2 - 5x - 6 = (x-6)(x+1) = 0$ has solutions $x = 6$ or $x = -1$. But x must be bigger than 2, therefore there is only one solution and it is $x = 6$.

Note that $x = -1$ satisfies the equation $\log_3 \frac{x(x-2)}{(x+2)} = 1$ because $\frac{-1(-1-2)}{(-1+2)}$ is positive. However, it does not satisfy the given equation $\log_3 x + \log_3 (x-2) - \log_3 (x+2) = 1$ for the simple reason that $\log_3(-1)$ and $\log_3(-1-2)$ are undefined.

Exercise 6

1. *Assume that b is a positive number. Use the fact that $\log_b 2 = 0.81$, $\log_b 5 = 1.88$ and $\log_b 7 = 2.28$, (rounded to 2 decimal places), to determine approximate values of the following*

   (a) $\log_b 35$ (b) $\log_b 70$ (c) $\log_b 1.4$

   (d) $\log_b 98$ (e) $\log_b \sqrt{56}$ (f) $\log_b 245$

2. *Determine x given that:*

   (a) $\log_3 x = 4$ (b) $\log_3(2x-1) = -2$ (c) $\left(\frac{1}{3}\right)^{2x} = 27$

   (d) $4^{(2x+1)} = 1$ (e) $2^{(4-3x)} = 16$ (f) $8^x = 32$

3. *Use properties of logarithms to expand the given expression as much as possible.*

   (a) $\log_3\left(27x^2y^3\right)$ (b) $\log_5\left(\frac{a^2b^4c}{125}\right)$ (c) $\log\left(100xyw^3\right)$

   (d) $\log_2\left(\frac{x^5y^3}{4w}\right)$ (e) $\log_3\left(\frac{27x^4y^3}{a^2z^5}\right)$ (f) $\log_2\left(64x^3y^2\sqrt{z}\right)$

   (g) $\ln\left(\frac{9}{25x^3\sqrt{y}}\right)$ (h) $\log_b\sqrt{\frac{5x+1}{x^3(x-1)}}$ (i) $\log_b \frac{y^3\sqrt{(y+3)}}{5(y^2+x^2)}$

4. *Condense the given expression into a single logarithm term.*

   (a) $\log(x+3) - \log(x-1) - 2\log(2x+5)$ (b) $\ln(3x-1) + \frac{1}{2}\ln(x-2) - 3\ln(4x+5) + \ln 7$

   (c) $\log_3(2x-1) + 3\log_3 5x - \frac{1}{4}\log_3\left(x^4+1\right)$ (d) $\log x + \log\left(x^2-9\right) - \log(x+3) - \log 5$

5. *Solve the following equations:*

   (a) $25 = 5^x$ (g) $12\left(\frac{7}{8}\right)^x = 0.9$ (m) $\log_2 x + \log_2(x+6) = 3$

   (b) $6^{x+3} = 9$ (h) $7(0.44)^{2x} = 1.21$ (n) $\ln(x+2) - \ln(x-1) = 3$

   (c) $2^{5x-2} = 7^x$ (i) $5^{2x+1} = 7^{x-2}$ (o) $\ln x - \ln(x-1) = 2$

   (d) $9 = 5(4^x)$ (j) $\log_4 x = 2$ (p) $2\log_3 x = 1 + \log_3(2x-3)$

   (e) $45 = e^{2x}$ (k) $\log_6(x+1) = 3$ (q) $\log_2 x + \log_2(3x-5) = 3$

   (f) $8 = 10e^{x-3}$ (l) $\log(2x+5) - \log(x-2) = 1$ (r) $\log_2(6x+1) - \log_2(x+2) = 2$

6. *The value of a certain machinery depreciates by 9% every year. This means that its value at the end of any year is 9% less than the value at the beginning of the year. It was worth $50000 when new. Complete the following table:*

# of years since it was new	0	1	2	3	x
Value of machinery in dollars	50000	50000 (0.91)			

How long will it take the value to drop below $10000?

7. *In food chemistry the formula*

$$M = 0.21\,(\log a - \log b)$$

is used in food processing to find the number of minutes M of heat treatment that a certain food should undergo at 250 degrees Celsius to reduce the probability of clostridium botulinum spores. "a" is the number of thousands of spores per can before heating and "b" is the number of thousands of spores per can after heating.

(a) *Find M for $a = 1$ and $b = 10^{-2}$.*

(b) *Find M for $a = 1.5$ and $b = 10^{-2}$.*

(c) *Given $M = 4$ minutes and $a = 3$, calculate b. Repeat for $M = 6$ and $a = 1.5$.*

8. *An experiment started with a culture of 1300 bacteria and it was predicted that t hours later the population P of the culture would be given by the equation*

$$P = 1300\left(4^{0.037t}\right).$$

After how many hours is the population predicted to double?

9. ***pH Scale**: The pH of a substance is determined by the formula*

$$pH = -\log\,[H+]$$

where $[H+]$ is the concentration of hydrogen ions in the substance measured in moles per liter.

(a) *The following table gives the moles of H+ per liter of several substance. Find the pH of each substance*

Substance	*[H+] (moles/L)*	*pH*
Eggs	1.6×10^{-8}	
Milk	4×10^{-7}	
Tuna Fish	9.55×10^{-7}	
Milk of Magnesia	3.16×10^{-7}	

(b) *The following table gives the pH of several substances. Find the concentration of H+ in moles per liter for each substance.*

Substance	*pH*	*[H+] (moles/L)*
Calcium hydroxides	11.4	
Vinegar	2.4	
Tomatoes	4.5	

10. *The amount A of radium-226 present after time t is given by the equation*

$$A = A_0 e^{-kt}$$

where A_0 is the amount present at time $t = 0$ and k is a constant. Suppose after 500 years, a sample of radium-226 has decayed to 80.4% of its original mass. Use this information to determine k, then find the half-life, (i.e. the number of years it takes half of the original mass to decay), of this chemical.

We have to mention the **change of base formula for logarithms**. To this end, suppose a, b, and x are positive numbers and $y = \log_a x$. Thus y is the exponent of a that equals x. We may write this as

$$x = a^y$$

It follows that $\log_b x = \log_b a^y$, and if we apply property (3) to the right hand side, we get

$$\log_b x = y \log_b a$$

Therefore $y = (\log_b x) \div (\log_b a)$. In other words

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

This is called the change of base formula. It enables us to calculate logarithms to bases that are not given on calculators. For example, if we are required to calculate $\log_{17} 46$, we simply note that

$$\log_{17} 46 = \frac{\log_{10} 46}{\log_{10} 17} = \frac{\log 46}{\log 17} = 1.351, \text{ rounded to 3 decimal places.}$$

Exercise 7 *Use the change of base formula to determine*

(a) $\log_7 19.5$ *(b)* $\log_{12} 0.76$ *(c)* $\log_5 88$