

Exponential Functions

You must have heard of the living organisms called amoebas. They "reproduce" by splitting into two. In other words, an amoeba divides into two parts and each part becomes an independent living amoeba. Suppose, for the sake of an illustration, each amoeba divides into two every day. If you start with one amoeba then 1 day later there will be 2 of them. Two days later, there will be 4. Three days later there will be 8; four days later there will be 16, and so on. Assume that there is an inexhaustible supply of nutrients for the amoebas and that there are no other creatures to eat them. Then their numbers will change as shown in the following table.

Number of days later	0	1	2	3	4	5	x
Number of amoebas	1	2	4	8	16	32	2^x

Denote this function by f . Its formula is $f(x) = 2^x$. Note that the variable x is an exponent. For this reason, f is called an exponential function. Since we get the images of f by raising 2 to different exponents, 2 is called the base of the exponential function, and we say that f is an exponential function to base 2. The number of amoebas increases with time x . This kind of increase is called an "exponential growth" and we say that the number of amoebas grows exponentially.

Some quantities decrease exponentially with time. The following is an example:

Example 1 *The resale value of a certain brand of cars decreases by about $\frac{1}{8}$ every year. More precisely, in any one year period, the value of such a car drops by $\frac{1}{8}$ of its value at the beginning of the period. For example, if its value was \$18,000 on January 1, 2006, then on January 1, 2007 the value had dropped to*

$$18,000 - 18,000 \times \frac{1}{8} = 18,000 \left(1 - \frac{1}{8}\right) = 18,000 \left(\frac{7}{8}\right) \text{ dollars}$$

To get a formula, it is advisable NOT to multiply 18,000 by $\frac{7}{8}$, (to get 15750). On January 1, 2007, the value had dropped by $\frac{1}{8}$ th of 18,000 $\left(\frac{7}{8}\right)$ to

$$18,000 \left(\frac{7}{8}\right) - 18,000 \left(\frac{7}{8}\right) \times \frac{1}{8} = 18,000 \left(\frac{7}{8}\right) \left(1 - \frac{1}{8}\right) = 18,000 \left(\frac{7}{8}\right)^2$$

On January 1, 2008, the value had dropped by $\frac{1}{8}$ th of 18,000 $\left(\frac{7}{8}\right)^2$ to

$$18,000 \left(\frac{7}{8}\right)^2 - 18,000 \left(\frac{7}{8}\right)^2 \times \frac{1}{8} = 18,000 \left(\frac{7}{8}\right)^2 \left(1 - \frac{1}{8}\right) = 18,000 \left(\frac{7}{8}\right)^3$$

Following the pattern, its value 4 years later will be 18,000 $\left(\frac{7}{8}\right)^4$ dollars. In general, its value x years later is 18,000 $\left(\frac{7}{8}\right)^x$ dollars. This information is summarized in the table below.

Number of years later	0	1	2	3	x
Value of car in \$	18,000	$18,000 \left(\frac{7}{8}\right)$	$18,000 \left(\frac{7}{8}\right)^2$	$18,000 \left(\frac{7}{8}\right)^3$	$18,000 \left(\frac{7}{8}\right)^x$

Denote this function by g . Its formula is $g(x) = 18000 \left(\frac{7}{8}\right)^x$, and it is an exponential function to base $\left(\frac{7}{8}\right)$. The numbers 18,000 $\left(\frac{7}{8}\right)^x$ get smaller as x increases, and we say that the resale value of the car decreases exponentially.

In general, an exponential function is characterized by having a formula in which the independent variable appears at least once as an exponent of some constant, called the base of the function. The most common ones, (which are on most scientific calculators) are the exponential function to base 10 with formula

$$f(x) = 10^x$$

and the exponential function to base e , (e is a number approximately equal to 2.71) with formula

$$g(x) = e^x.$$

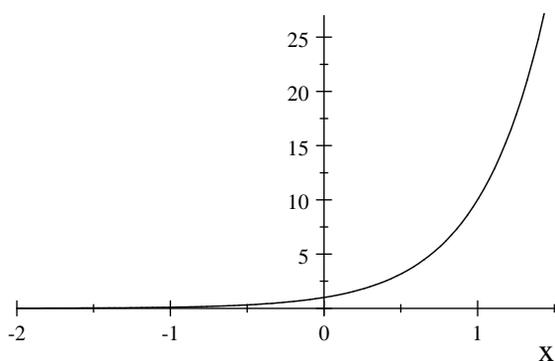
Use a calculator to complete the following tables:

x	1.3	4.22	3.05	0.6	-0.8	-1.7	-2.32	0.07	5.3
$f(x) = 10^x$									

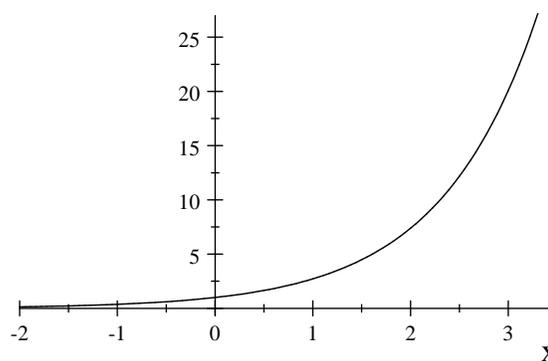
(1)

x	-1.7	3.55	4.17	0.8	-0.9	-1.68	-2.05	0.09	5.6
$g(x) = e^x$									

The graphs of $f(x) = 10^x$ and $g(x) = e^x$ are given below 10^x

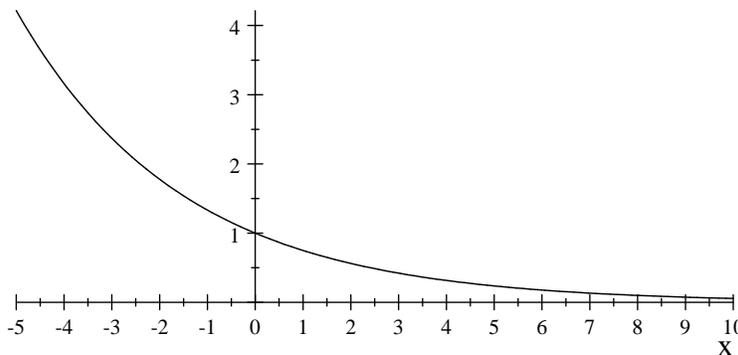


Graph of $f(x) = 10^x$



Graph of $g(x) = e^x$

As their graphs show, the values of f and g increase as x increases. This is because their bases, i.e. 10 and e are bigger than 1. The values of an exponential function to a base that is smaller than 1 decrease as x increases. The graph of $h(x) = \left(\frac{3}{4}\right)^x$ illustrates this point. $\left(\frac{3}{4}\right)^x$



Exercise 2

1. The number of bacteria $N(t)$ in a certain culture t hours after 8:00 AM is given by $N(t) = 500(5)^{t/2}$. Find the number of bacteria in the culture at 10:00 AM and at 11:00 AM.
2. **Company growth** The growth of a company can be described by the equation

$$N = 500(1.02)^{0.7t}$$

where t is the number of years the company has been in existence and N is the number of employees. What is the number of employees when $t = 0$, (this is the number of employees the company has when it starts), and the number when $t = 5$?

3. The value of a car depreciates by 11% every year. (This means that its value at the end of any year is 11% less than the value at the beginning of the year.) If it was worth \$20,000 when new. Complete the following table.

# of years since it was new	0	1	2	3	4	x
Value of car in dollars	20000	20000 (0.89)				

What will be the value of the car 10 years after being purchased?

4. If a certain make of automobile is purchased for A dollars, its trade-in value $V(t)$ at the end of t years is given by the formula $V(t) = 0.78A(0.85)^{t-1}$. Given that such an automobile was purchased for \$24,000, find its trade-in value, to the nearest dollar, at the end of 5 years.
5. **Population growth** Suppose the formula for the growth of the population of a city for the next 10 years is given by the formula

$$P = 30,000(2.5)^{0.03t}$$

where P_0 is the population of the city at the present time and P is the population t years from now. What is the population predicted to be 10 years from now? Round off your answer to the nearest whole number.

6. A radioactive chemical element has a nucleus that is unstable and will change, or decay. The half-life of a radioactive chemical is defined as the time it takes for half the total number of radioactive nuclei to decay. Technetium-99m is a radioactive substance used to diagnose brain, thyroid liver and kidney diseases. This radioactive substance has a half life of 6 hours. If there are 200 milligrams of this technetium-99m, how much will there be in

$$(a) 6 \text{ hours?} \quad (b) 24 \text{ hours?} \quad (c) t \text{ hours?}$$

7. Suppose it takes one day for a certain strain of bacteria to reproduce by dividing in half. If 100 cells of bacteria are present to begin with then the total cell count after x days is

$$b = 100(2^x)$$

- (a) Find the total number of bacteria cells presence after $x = 1, 2, 3,$ and 4 days.
 (b) By how much does the number of bacteria increase between day 6 and day 8?

8. **Global travel spending** Global spending S for travel and tourism (in billion of dollars) for the years 1990-2020 can be approximated by the equation

$$S = 43.3t^{4/5}$$

where t is the number of years past 1980. What is the estimated global spending on travel for the year 2017?

9. **Transportation** The percent p of paved roads and streets in the United States can be approximated with the equation

$$p = 6.61t^{0.56}$$

where t is the number of years past 1940.

- (a) Does this equation estimate a greater percent change during the decade of the 1970s or during the decade from 2000 to 2010? What might explain this?
 (b) When can you be certain this equation is no longer valid?