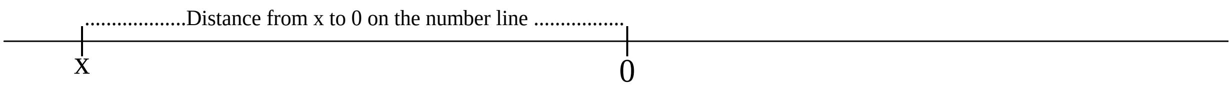
Absolute Value of a Number

To visualize the absolute value of a given number x, draw a number line with 0 in its middle and plot x on the line. Then its absolute value is, (by agreement), its distance from 0 to x. It is denoted by $|x|$.



Example 1

1. $|5| = 5$, $|-4| = 4$ *and* $|-3.7| = 3.7$
2. *Since the distance from 0 to 0 on the number line is zero, the absolute value of* 0 *is* $|0| = 0$.

By agreement, distances cannot be negative, therefore the absolute value of a number cannot be negative. Actually every number, except 0, has a positive absolute value; 0 is the only number whose absolute value is zero.

It should be clear from the number line that the absolute value of a positive number x is x itself. Thus: $|0.88| = 0.88$, $|2.8| = 2.8$, $|9| = 9$. On the other hand, the absolute value of a negative number y is obtained by "throwing" away the negative sign, which amounts to taking the negative of y. For example,

$$|-6.3| = 6.3 \text{ and since } 6.3 = -(-6.3), \text{ we may write this as } |-6.3| = -(-6.3).$$

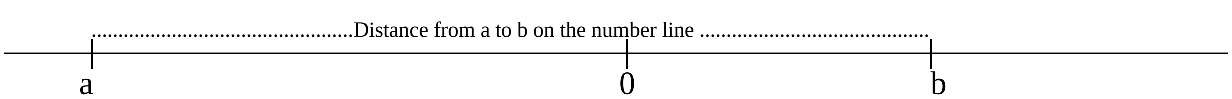
These observations suggest the following formal definition of the absolute value of a number that makes no mention of distances:

Definition 2 *The absolute value of a number x is denoted by $|x|$ and is given by*

$$|x| = \begin{cases} x & \text{if } x \text{ is not negative} \\ -x & \text{if } x \text{ is negative} \end{cases}$$

Distance From a Number a to an Arbitrary Number b

We generalize by consider the distance from a given number a to another number b which need not be 0. Since the distance from a number x to 0 was denoted by $|x|$, which can also be written as $|x - 0|$, the distance from a to b should be written as $|a - b|$, and this is precisely what we plan to do. More precisely, given arbitrary numbers a and b, the absolute value of $a - b$ is denoted by $|a - b|$ and it is the distance from a to b on the number line.



In particular

1. $|7 - 12|$ is the distance between 7 and 12 on the number line, which is 5.

2. $|-8.7-2|$ is the distance between -8.7 and 2 which is 10.7.

3. $|6+3|$, which we may also write as $|6-(-3)|$, is the distance between 6 and -3 on the number line. It is 9.

A definition of $|a-b|$ that does not mention distances is the following:

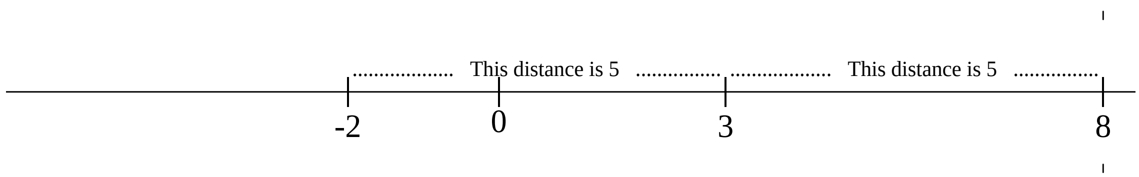
Definition 3 *If a and b are given numbers then*

$$|a-b| = \begin{cases} a-b & \text{if } a-b \text{ is not negative} \\ -(a-b) & \text{if } a-b \text{ is negative} \end{cases}$$

Example 4

1. *To find x given that $|x-3|=5$.*

   We must find numbers x whose distance from 3 is 5. Draw the number line and plot 3. If you start from 3 and move 5 steps to the right you land at 8, while 5 steps to the left land you at -2. Therefore x must be -2 or 8.



   If you prefer using the formal definition of an absolute value, here is one way to go about it: According to the definition, $|x-3| = x-3$ if $x-3$ is not negative. Then the given equation becomes

   $$x-3=5 \text{ which may be solved to get } x=8$$

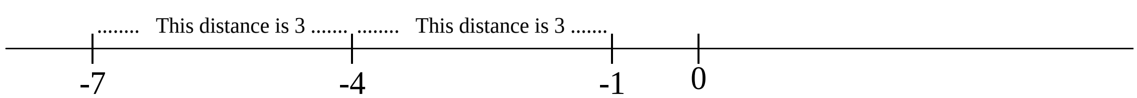
   On the other hand, if $x-3$ is negative them $|x-3| = -(x-3)$ and the given equation becomes

   $$-(x-3)=5 \text{ which may also be written as } x-3=-5. \text{ Now solving gives } x=-2$$

   Therefore $x=8$ or -2

2. *To find x given that $|x+4|=3$.*

   We start by writing the given equation as $|x-(-4)|=3$. Now it is clear that we must find numbers x whose distance from -4 is 3. Draw the number line and plot the number -4. If you start from -4 and move 3 steps to the left, you land at -7. If you move 3 steps to the right you land at -1. Therefore x is -7 or -1.



3. *To find x if $|x-2|>9$.*

   We have to find all the numbers x whose distance from 2 is bigger than 9. Draw the number line, plot the number 2 then look for the numbers x whose distance from 2 is bigger than 9. If you look to the left of 2 you should conclude that x must be smaller than -7. If you look to the right of 2 you should conclude that x is bigger than 11. Therefore $x<-7$ or $x>11$.

If you prefer using the formal definition, you would argue that $|x-2| = x-2$ *if* $(x-2)$ *is not negative, therefore the given inequality would be*

$$x - 2 > 9 \text{ which you would solve to get } x > 11$$

On the other hand, $|x-2| = -(x-2)$ *if* $(x-2)$ *is negative. Then the given inequality would be*

$$-(x-2) > 9 \text{ or } x - 2 < -9. \text{ Solving gives } x < -7$$

Therefore $x < -7$ *or* $x > 11$.

4. *What is* x *if* $|x+3| = |x+11|$?.

   We may answer this question using the number line as follows: Write the equation as

   $$|x-(-3)| = |x-(-11)|.$$

   Now we see that we need a number x *whose distance from* -3 *is equal to its distance from* -11. *Clearly, it is the number half-way between* -3 *and* -11 *on the number line, which is* -7.

   If you prefer a purely algebraic solution, here is one: Either $|x+3| = x+3$ *(if the expression is not negative) or* $|x+3| = -(x+3)$ *if the expression is negative. Likewise, either* $|x+11| = x+11$ *or* $|x+11| = -(x+11)$. *Therefore, if* $|x+3| = |x+11|$ *then either* $x+3 = x+11$ *(if the two expressions have the same sign), or* $x+3 = -(x+11)$ *(if one of them is not negative and the other one is negative). But it is impossible to have* $x+3 = x+11$ *because it would imply that* $3 = 11$ *which is not true. Therefore the only possibility is* $x+3 = -(x+11)$ *which we may re-arrange to get* $2x = -14$, *then solve to get* $x = -7$.

Exercise 5

1. *What is* x *if:*
   (a) $|x| = 9$
   (b) $|x| = 16.8$
   (c) $|x| = -1$

2. *What are all the possible values of* x *if:*
   (a) $|x| \leq 7$ *(i.e. the absolute value of* x *is less than or equal to 7)?*
   (b) $|x| > 3$ *(i.e. the absolute value of* x *is bigger than 7)?*
   (c) $|x-3| < 4$?
   (d) $|x-2| = 1$?
   (e) $|x-1| = |x+5|$?
   (f) $|x+4| = |x+12|$
   (g) $|x-1| = |2x+7|$ *(Hint: use the formal definition of an absolute value.)*