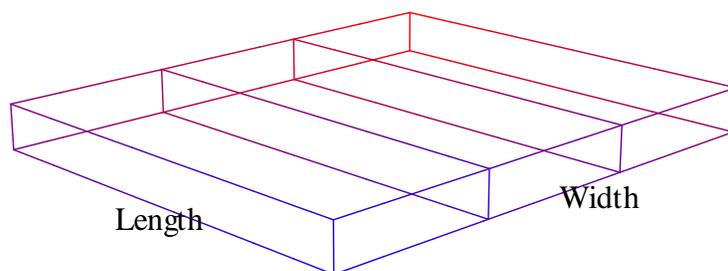


Quadratic Functions

The following is a problem that leads directly to a quadratic function:

Example 1 A farmer has 500 yards of fencing to construct three equal enclosures, shown below (not drawn to scale), for his animals.



What is the largest possible area he can enclose? Before we answer this question, complete the following table to see how the length, width and area of such an enclosure are related:

Length	30	45	60	70	85	100	x
Width	190		130	110		50	
Area	5700	7200	7800			5000	$250x - 2x^2$

To get a function we form a table with two rows as below

Length	30	45	60	70	85	100	x
Area	5700	7200	7800			5000	$250x - 2x^2$

Denote it by A , (for total area). Then

$$A(x) = 250x - 2x^2$$

This is an example of a quadratic function. The answer to the question "What is the largest possible area he can enclose" is given later in this section after a discussion of graphs of quadratic functions.

A general quadratic function is any function f with a formula of the form

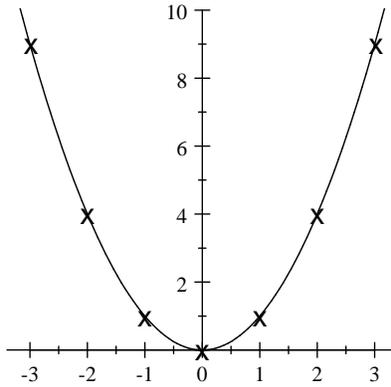
$$f(x) = ax^2 + bx + c$$

where a , b and c are constants. Its graph is called a parabola.

For the quadratic function $A(x) = 250x - 2x^2$ in Example 1, $a = -2$, $b = 250$ and $c = 0$.

The simplest quadratic function is $f(x) = x^2$ in which $a = 1$, $b = 0$, and $c = 0$. To plot its graph, one constructs a table with a sample of its values and plots the corresponding points. One then joins the points with a smooth curve as shown below.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9



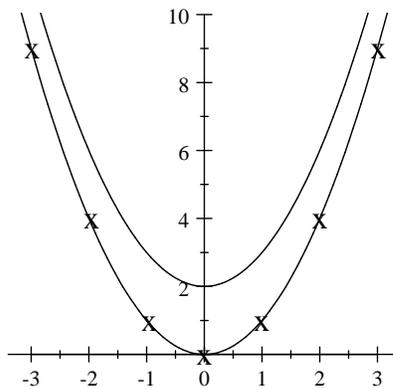
The graphs of other quadratic functions are obtained by modifying this basic graph. We introduce them in stages:

1. The graph of a function whose formula is obtained by adding a positive constant to x^2 .

Example 2 Consider $g(x) = x^2 + 2$. Sample values of f and g are given in the table below

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 2$	11	6	3	2	3	6	11

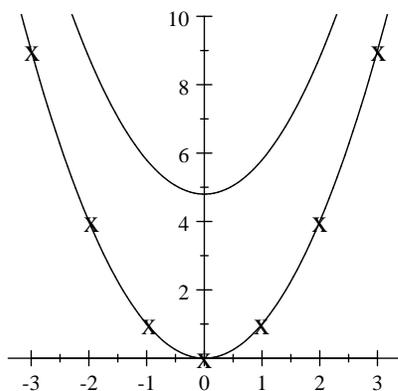
The value of g at any number x is 2 units bigger than the value of f at x . For example, the value of g at -3 is 11 which is 2 units bigger than 9, (the value of f at -3). Therefore the graph of g is 2 units above the graph of f .



In other words, the graph of g is obtained by sliding the graph of f two units up. The more common term used is "translating vertically". Thus the graph of g is obtained by vertically translating the graph of f two units up.

In general, if b is a positive number then to get the graph of $h(x) = x^2 + b$, simply is obtained by vertically translating the graph of $f(x) = x^2$ by b units up. The graphs of $f(x) = x^2$ and $h(x) = x^2 + 4.8$ are

shown below.

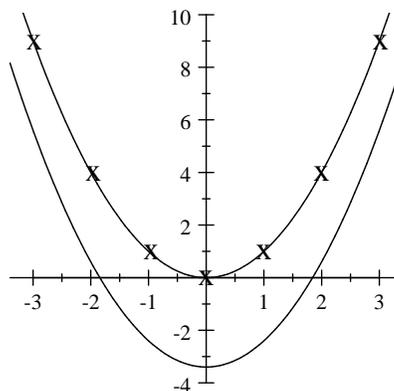
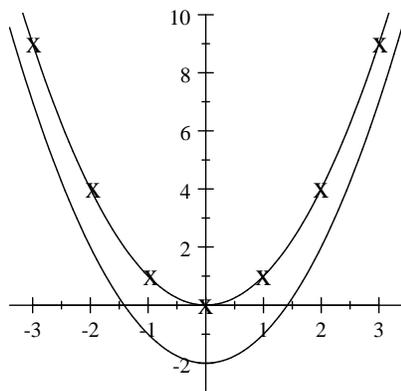


2. The graph of a function whose formula is obtained by adding a negative constant to x^2 . Examples:

$$k(x) = x^2 - 2, \text{ obtained by adding } -2 \text{ to } x^2$$

$$p(x) = x^2 - 3.4, \text{ obtained by adding } -3.4 \text{ to } x^2$$

are $k(x) = x^2 - 2$ (obtained by adding -2 to x^2), and $p(x) = x^2 - 3.4$ (obtained by adding -3.4 to x^2). Their graphs are shown below. In general, if c is a positive constant then the graph of $v(x) = x^2 - c$ is obtained by simply translating the graph of $f(x) = x^2$ down c units.



Graphs of $f(x) = x^2$ and $k(x) = x^2 - 2$ Graphs of $f(x) = x^2$ and $p(x) = x^2 - 3.4$

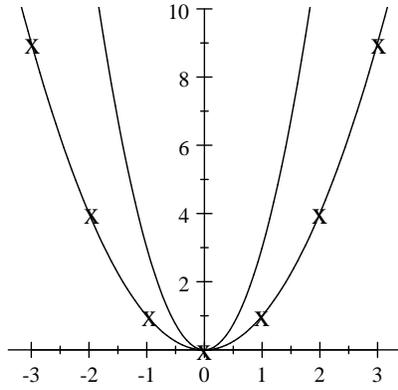
3. The graph of a function whose formula is obtained by multiplying x^2 by a positive constant. Examples:

$$w(x) = 3x^2, \text{ obtained by multiplying } x^2 \text{ by } 3$$

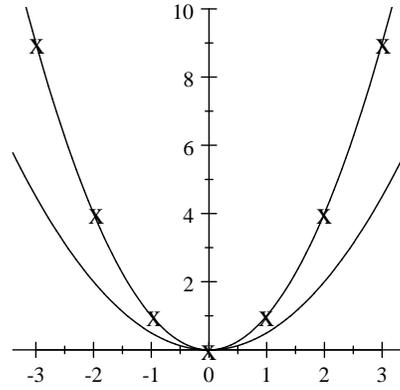
$$v(x) = \frac{1}{2}x^2, \text{ obtained by multiplying } x^2 \text{ by } \frac{1}{2}$$

The resulting graph is steeper than the graph of f , if the constant is bigger than 1, or less steep if the

constant is a fraction between 0 and 1.



Graphs of $f(x) = x^2$ and $w(x) = 3x^2$



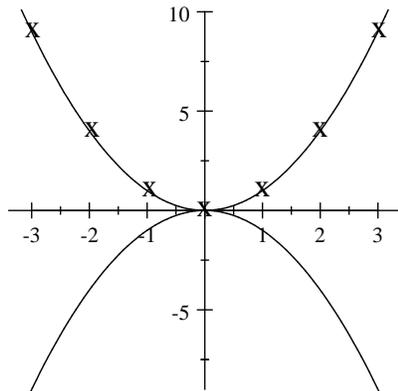
Graphs of $f(x) = x^2$ and $v(x) = \frac{1}{2}x^2$

4. The graph of a function whose formula is obtained by multiplying x^2 by a negative constant. Examples:

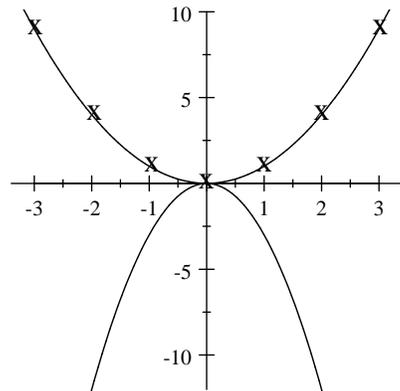
$$t(x) = -x^2, \text{ obtained by multiplying } x^2 \text{ by } -1$$

$$r(x) = -3x^2, \text{ obtained by multiplying } x^2 \text{ by } -3$$

The resulting graphs open down as shown below



Graphs of $f(x) = x^2$ and $t(x) = -x^2$



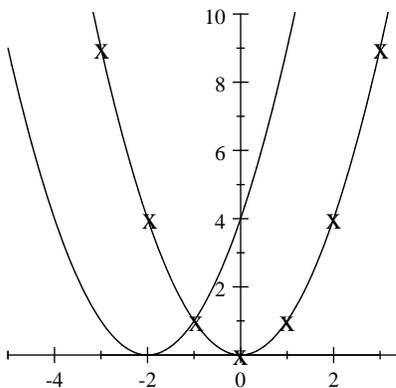
Graphs of $f(x) = x^2$ and $q(x) = -3x^2$

5. The graph of a function whose formula is obtained by adding a constant to the variable x then square.

Example 3 Consider the function p with formula $p(x) = (x + 2)^2$, obtained by adding 2 to x then square. Sample values of $f(x) = x^2$ and p are given in the table below

x	-5	-4	-3	-2	-1	0	1	2	3	4
x^2	25	16	9	4	1	0	1	4	9	14
$(x + 2)^2$	9	4	1	0	1	4	9	16	25	36

Their graphs are shown below

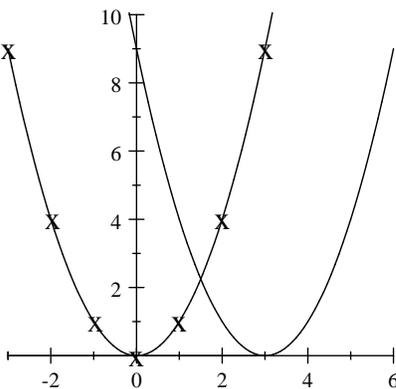


Graphs of $f(x) = x^2$ and $p(x) = (x + 2)^2$

Note that f has the same value at 0 as p has at -2 , it has the same value at 1 as p has at -1 , in general f has the same value a number b as p has two steps back. Therefore the graph of p is obtained by sliding the graph of f , (along the horizontal axis), two units to the left.

Example 4 Consider the function q with formula $q(x) = (x - 3)^2$, obtained by adding -3 to x then square. Sample values of $f(x) = x^2$ and q are given in the table below

x	-3	-2	-1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
$(x - 3)^2$	36	25	16	9	4	1	0	1	4	9

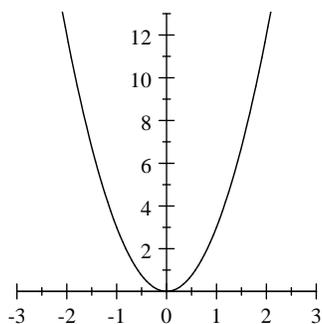


Graphs of $f(x) = x^2$ and $q(x) = (x - 3)^2$

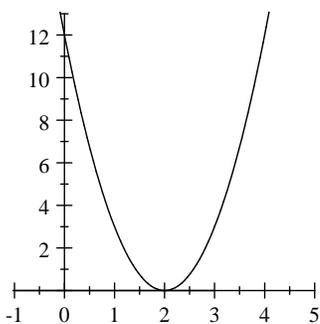
Note that f has the same value at 0 as q has at 3, it has the same value at 1 as q has at 4, in general f has the same value a number b as q has three steps forward. Therefore the graph of q is obtained by sliding the graph of f , (along the horizontal axis), three units to the right.

6. The graphs of the other quadratic functions are obtained by performing two or more of the above procedures. We illustrate with examples.

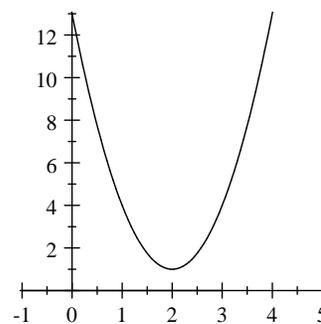
- (a) Say we have to sketch the graph of $f(x) = 3(x - 2)^2 + 1$. We start with the sketch of $3x^2$, translate it 2 units to the right to get the graph of $3(x - 2)^2$, then slide it 1 unit up to get the graph of $3(x - 2)^2 + 1$. The steps are shown below.



Graph of $3x^2$

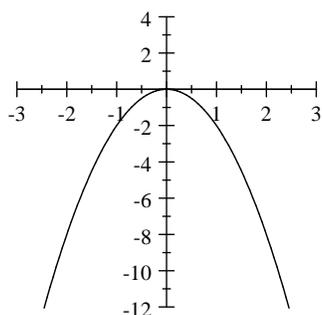


Graph of $3(x - 2)^2$

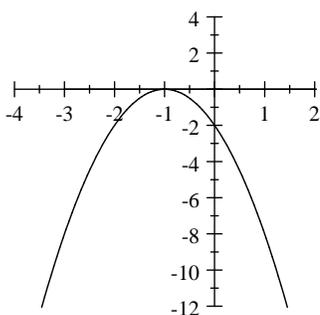


Graph of $3(x - 2)^2 + 1$

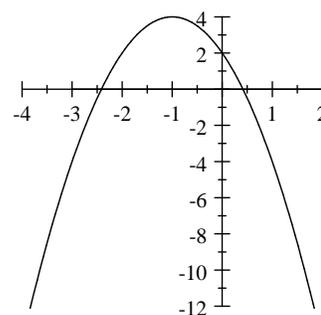
- (b) To sketch the graph of $g(x) = -2(x + 1)^2 + 4$ we start with a sketch of $-2x^2$, translate it 1 unit to the left then slide it up 4 units



Graph of $-2x^2$



Graph of $-2(x + 1)^2$

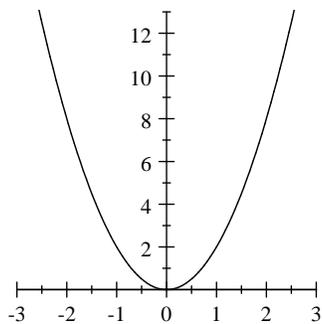


Graph of $-2(x + 1)^2 + 4$

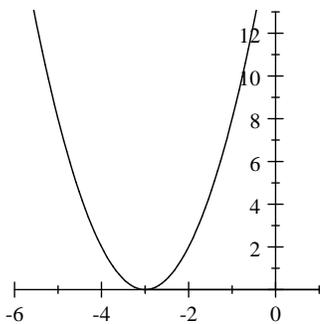
- (c) To sketch the graph of $u(x) = 2x^2 + 12x + 3$, we first complete squares as follows:

$$2x^2 + 12x + 3 = 2(x^2 + 6x) + 3 = 2[(x + 3)^2 - 9] + 3 = 2(x + 3)^2 - 18 + 3 = 2(x + 3)^2 - 15$$

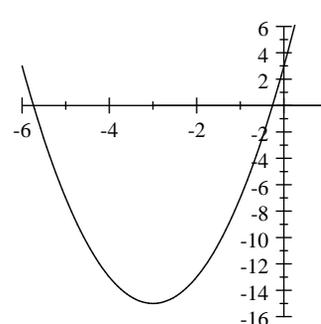
Its graph opens up. To sketch it, start with the graph of $2x^2$, translate it 3 units to the left then move it down 15 units.



Graph of $2x^2$



Graph of $2(x + 3)^2$



Graph of $2(x + 3)^2 - 15$

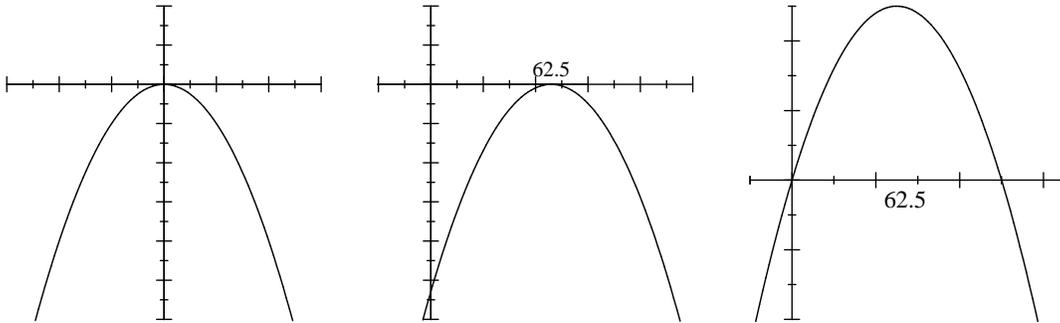
Now we can solve the farmer's problem. We have to figure out the largest value of

$$A(x) = -2x^2 + 250x = -2(x^2 - 125x)$$

When we complete squares, we get

$$A(x) = -2 \left[\left(x - \frac{125}{2}\right)^2 - \left(\frac{125}{2}\right)^2 \right] = -2 \left(x - \frac{125}{2}\right)^2 + 2 \left(\frac{125}{2}\right)^2$$

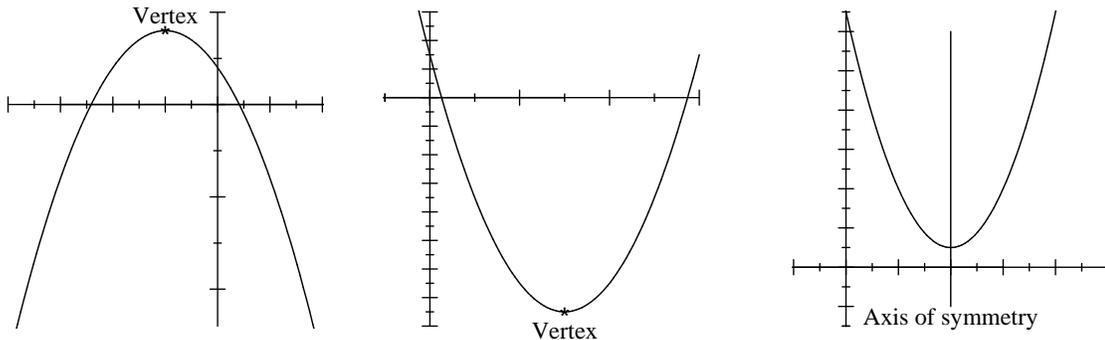
The graph of $A(x) = -2 \left(x - \frac{125}{2}\right)^2 + 2 \left(\frac{125}{2}\right)^2$ opens down. To sketch it, start with $-2x^2$, translate it $\frac{125}{2}$ units to the right then move it $2 \left(\frac{125}{2}\right)^2$ units up. It is the section of the graph of A between $x = 0$ and $x = 125$ that is relevant to the farmer's problem.



The highest point on the graph is at $x = \frac{125}{2} = 62.5$. Therefore the maximum area he can enclose is $A\left(\frac{125}{2}\right)$ which is $\frac{15625}{2} = 7812.5$ square yards when the enclosure has length 62.5 yards.

Some more terms that apply to quadratic functions

- The lowest point on a parabola, if it opens up, or the highest point on the parabola if it opens down, is called the **VERTEX** of the graph. The vertical line that "slices the graph into two look-alike halves" is called the **AXIS OF SYMMETRY** of the graph.



Example 5 The graph of the function $A(x) = -2x^2 + 250x = -2 \left(x - \frac{125}{2}\right)^2 + 2 \left(\frac{125}{2}\right)^2$ we constructed to determine the largest area the farmer in Example 1 can enclose has vertex $\left(\frac{125}{2}, \frac{15625}{2}\right)$ and axis of symmetry $x = \frac{125}{2}$.

Example 6 The graph of $u(x) = 2x^2 + 12x + 3 = 2(x + 3)^2 - 15$ has vertex $(-3, -15)$ and axis of symmetry $x = -3$.

As these two examples show, if a quadratic function f is written in the form $f(x) = d(x - e)^2 + k$, (d , e and k are constants), then its vertex is the point (e, k) and its axis of symmetry is the line $x = e$. Thus the axis of symmetry is the number that makes the squared term $d(x - e)^2$ zero. That number is also the x -component of the vertex. The y -component of the vertex is the constant term k that is added to, (or subtracted from), the squared term $d(x - e)^2$.

Example 7 To determine the vertex and axis of symmetry for a general quadratic function $f(x) = ax^2 + bx + c$, one must first complete squares:

$$\begin{aligned} f(x) &= ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x\right] + c = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a} \end{aligned}$$

Therefore the axis of symmetry is the line $y = -\frac{b}{2a}$ and the vertex is $\left(-\frac{b}{2a}, \frac{-b^2 + 4ac}{4a}\right)$.

Exercise 8

1. Match each function with its graph then give the vertex and axis of symmetry of the graph.

(a) $(x - 3)^2$

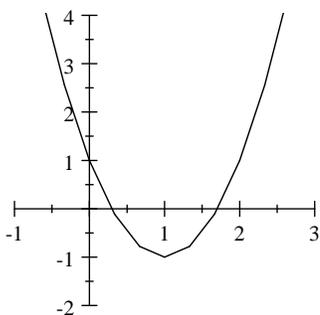
(b) $(x + 2)^2$

(c) $-3x^2$

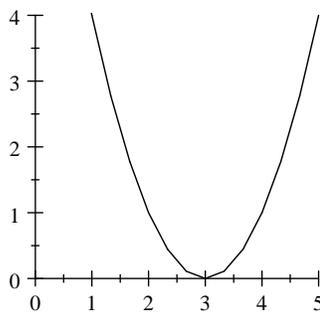
(d) $2(x - 1)^2 - 1$

(e) $-x^2 + 4x - 1$

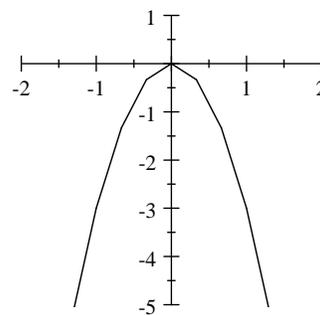
(f) $-3x^2 - 6x + 2$



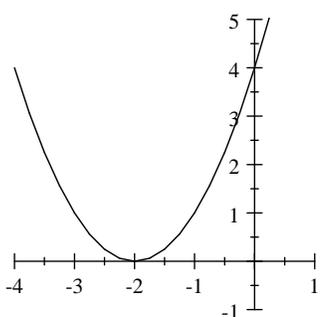
(i)



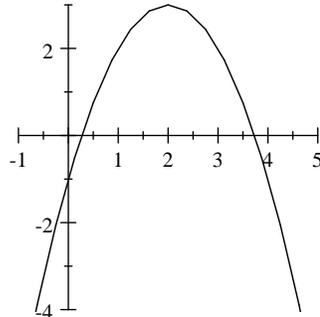
(ii)



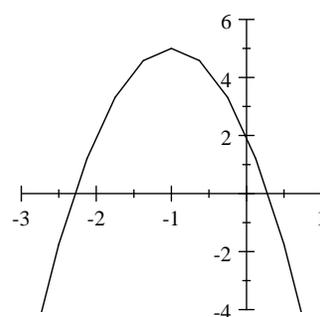
(iii)



(iv)



(v)



(vi)

2. For each quadratic function, determine the vertex and axis of symmetry of its graph then sketch the graph. (In this exercise, sketching a graph means giving its expected shape with the axis of symmetry and vertex clearly indicated.)

(a) $f(x) = -\frac{2}{3}x^2 - 3$

(b) $g(x) = x^2 + 5$

(c) $h(x) = -(x - 3)^2 + 2$

(d) $u(x) = x^2 + 4x$

(e) $w(x) = \frac{1}{2}(x + 3)^2 - 4$

(f) $v(x) = 2(x - 5)^2 + 3$

3. What is the equation of the quadratic function whose graph is shown below?

