Inverse of a function

Before introducing the inverse of a function, we need to introduce the idea of a one-to-one function. It is a function f that matches distinct elements u and v in its domain with different images $f(u)$ and $f(v)$. Stated differently, f is one-to-one if any two different numbers u and v in its domain have different images $f(u)$ and $f(v)$. Therefore, the only way images $f(u)$ and $f(v)$ can be the same is if $u = v$.

Example 1 *Take the function f with formula $f(x) = 3x + 1$. If $f(u) = f(v)$ then $3u + 1 = 3v + 1$. But this implies that $3u = 3v$, i.e. that $u = v$. Therefore f is one-to-one.*

Example 2 *The function g with formula $g(x) = x^2 + 1$ is not one-to-one because we can find different numbers in its domain that have the same image. For example, -2 and 2 have the same image 5.*

Turning to inverses, only a one-to-one function can have an inverse. The inverse of such a function f is the table you get when you swap the two rows, (or the two columns if the function is represented by a table with two columns), in the table for f. It is denoted by f^{-1}. Thus the domain of f^{-1} is the range of f and its range is the domain of f.

Example 3 *Consider f with formula $f(x) = 3x + 1$. It is displayed as a table below.*

x	1	2	3	4.5	x	$\frac{1}{3}(y-1)$
$f(x)$	4	7	10	14.5	$3x+1$	y

Its inverse f^{-1} is displayed as a table below.

y	4	7	10	14.5	y
$f^{-1}(y)$	1	2	3	4.5	$\frac{1}{3}(y-1)$

Its formula is $f^{-1}(y) = \frac{1}{3}(y-1)$. We chose to use y, instead of x, in the formula for f^{-1} purely for convenience.

Evaluate $f^{-1}(f(2))$ and $f^{-1}(f(4.5))$. Why must $f^{-1}(f(x)) = x$ for all x in the domain of f?

Example 4 *The function $g(x) = 2x - 1$, with table below*

x	3	4	5.2	7	x	$\frac{1}{2}(y+1)$
$g(x)$	5	7	9.4	13	$2x-1$	y

has inverse

y	5	7	9.4	13	y
$g^{-1}(y)$	3	4	5.2	7	$\frac{1}{2}(y+1)$

Evaluate $g(g^{-1}(9.4))$ and $g(g^{-1}(5))$. Why must $g(g^{-1}(y)) = y$ for all y in the range of g?

One way of determining the formula for the inverse f^{-1} given the formula for f is to write down the equation $f(x) = y$ then solve for x in terms of y. For example, let f have formula $f(x) = 4x - 3$. Write $4x - 3 = y$ and solve for x. You should get $x = \frac{1}{4}(y + 3)$, therefore

$$f^{-1}(y) = \frac{1}{4}(y + 3)$$

In Examples 3 and 4, you were asked to note that an inverse f^{-1} of a function f satisfies the conditions $f^{-1}(f(x)) = x$ for every number x in the domain of f and $f(f^{-1}(y)) = y$ for every number y in the range of f. We will use these properties repeatedly in a number of instances. We already applied them to radicals when we stated that

$$\sqrt{x^2} = x \text{ if } x \text{ is not negative} \quad \text{and} \quad \left(\sqrt{y}\right)^2 = y \text{ if } y \text{ is not negative}$$

$$\sqrt[3]{x^3} = x \quad \text{and} \quad \left(\sqrt[3]{y}\right)^3 = y$$

In general, if the nth root $\sqrt[n]{y}$ of a number y is defined then $\left(\sqrt[n]{y}\right)^n = y$ and $\sqrt[n]{y^n} = y$

To defend the claim that $\sqrt{x^2} = x$, take $f(x) = x^2$, $x \geq 0$. It has an inverse, namely $f^{-1}(x) = \sqrt{x}$. Therefore

$$f^{-1}(f(x)) = x \text{ which translates into } \sqrt{x^2} = x$$

The identity $f(f^{-1}(y)) = y$ translates into $\left(\sqrt{y}\right)^2 = y$

Exercise 5 *Calculate the inverse of each function:*

1. $f(x) = 5x + 1$
2. $g(x) = 4 - 3x$
3. $h(x) = x^2 + 3$ *with the domain consisting of all the positive numbers.*