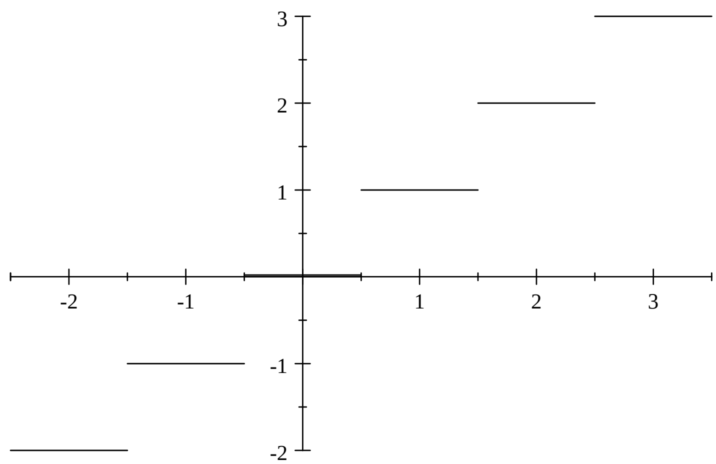
# Limits From Below and Limits From Above

The following example illustrates the idea of a limit from below and a limit from above.

**Example 1** *Imagine rounding off a number to the nearest whole number. This means that you replace a given number by the integer nearest to it on the number line. For example, since* $5.184$ *is between* $5$ *and* $6$*, but it is closer to* $5$*, you round it off to* $5$*. A number in the middle of two integers will, (by agreement), be rounded off to the larger of the two integers. Thus* $3.5$ *is rounded off to* $4$ *and* $-16.5$ *is rounded off to* $-16$*. Let* $f$ *be the function that assigns the nearest whole number to a given number* $x$ *as described. Part of its graph is given below.*




*Take a number like* $1.5$ *in the middle of two integers. Every number* $x$ *that is close to 1.5 and is also to the left of 1.5 on the number line, (examples are* $1.4$*,* $1.488$*, and* $1.4999$*), gives a value* $f(x)$ *close to* $1$ *(actually equal to* $1$*). For this reason, we say that* $f$ *has limit* $1$ *as* $x$ *approaches* $1.5$ *from below,(or from the left), and write* $\lim_{x \to 1.5^-} f(x) = 1$.

*On the other hand, numbers* $x$ *close to* $1.5$*, but to the right of* $1.5$*, (example,* $1.51, 1.5003$*, and* $1.500008$*), give values* $f(x)$ *close to* $2$*, (actually equal to* $2$*). We say that* $f$ *has limit* $2$ *as* $x$ *approaches* $1.5$ *from above, (or from the right), and write* $\lim_{x \to 1.5^+} f(x) = 2$.

The following are the general definitions of limits from below/above:

**Definition 2** *A function* $f$ *has limit* $l$ *as* $x$ *approaches a given number* $c$ *from below if numbers* $x$ *that are; (i)* ***to the left of*** $c$*, and (ii)* ***are close to*** $c$*, give values* $f(x)$ *that are close to* $l$*. The number* $l$ *is called the limit of* $f$ *at* $c$ *from below, or the left limit of* $f$ *at* $c$*, and is denoted by* $\lim_{x \to c^-} f(x)$*.*

As we pointed out before giving definition **??**, numbers close to $l$ are in an interval $(l - \varepsilon, l + \varepsilon)$ for some suitable positive number $\varepsilon$. Likewise, numbers which are close to $c$ and are also to the left of $c$ are in some interval $(c - \delta, c)$ for some suitable positive number $\delta$. This suggests the following more precise definition of a limit from the left, (or from below).

**Definition 3** *A function* $f$ *has limit* $l$ *as* $x$ *approaches a number* $c$ *from below if given any positive number* $\varepsilon$*, we can find a positive number* $\delta$ *such that* $f(x) \in (l - \varepsilon, l + \varepsilon)$ *for all* $x \in (c - \delta, c)$*. Alternatively, given any* $\varepsilon > 0$ *we can find a* $\delta > 0$ *such that* $|f(x) - l| < \varepsilon$ *if* $c - \delta < x < c$*.*

**Definition 4** *A function* $f$ *has limit* $m$ *as* $x$ *approaches a given number* $c$ *from above if numbers* $x$ *that are; (i)* ***to the right of*** $c$*, and (ii)* ***are close to*** $c$*, give values* $f(x)$ *that are close to* $m$*. The number* $m$ *is called the limit of* $f$ *at* $c$ *from above, or the right limit of* $f$ *at* $c$*, and is denoted by* $\lim_{x \to c^+} f(x)$*.*

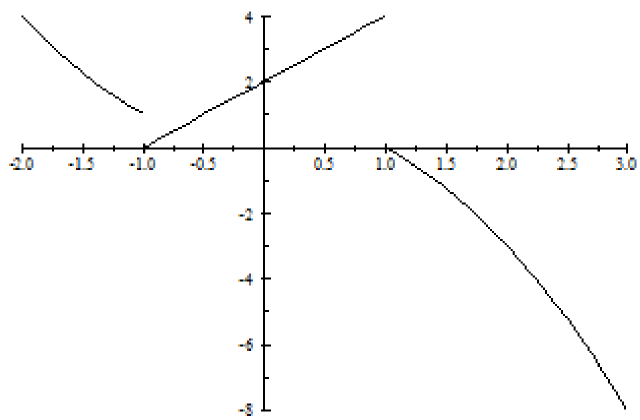
Since numbers which are close to $c$ and are also to the right of $c$ are in some interval $(c, c+\delta)$ for some suitable positive number $\delta$, a more precise definition of a limit from the right (or from above) is the following:

**Definition 5** *A function $f$ has limit $l$ as $x$ approaches a number $c$ from above if given any positive number $\varepsilon$, we can find a positive number $\delta$ such that $f(x) \in (l-\varepsilon, l+\varepsilon)$ for all $x \in (c, c+\delta)$. Alternatively, given any $\varepsilon > 0$ we can find a $\delta > 0$ such that $|f(x) - l| < \varepsilon$ if $c < x < c + \delta$.*

**Example 6** *Consider the function $g$ with formula*

$$g(x) = \begin{cases} x^2 & if \quad -2 \leq x \leq -1 \\ 2x+2 & if \quad -1 < x \leq 1 \\ 1-x^2 & if \quad 1 < x \leq 3 \end{cases}$$

*Its graph is given below.*




*Take a number like $c = 1$. The graph and formula for $g$ suggest that $\lim_{x\to 1^+} g(x) = 0$ and $\lim_{x\to 1^-} g(x) = 4$. In the case of $c = -1$, the graph and formula suggest that $\lim_{x\to -1^+} g(x) = 0$ and $\lim_{x\to -1^-} g(x) = 1$.*

**Remark 7** *If a function $f$ has limit $l$ as $x$ approaches a number $c$ then all the numbers $x$ close to $c$, (whether to the left or to the right of $c$), give values $f(x)$ close to $l$. Therefore $f$ must have limit $l$ as $x$ approaches $c$ from below and from above. In other words,*

$$\lim_{x\to c^-} f(x) = l = \lim_{x\to c^+} f(x)$$

**Remark 8** *It follows from Remark 7 that if the left limit of a function $g$ at a point $c$ is different from its right limit, (in other words, if $\lim_{x\to c^-} g(x) \neq \lim_{x\to c^+} g(x)$), then $g$ has no limit at $c$.*