Functions and Function Notation

Statements that one variable quantity is a function of, (i.e. depends on), another variable quantity are quite common. Here are some:

1. The resale value of a used car is a function of, (among other factors), its age. Generally, the older it is , the lower the price.
2. The demand for gas is a function of, (i.e. it depends on), its price. When the price falls, demand goes up. When the price is high, fewer people buy. (Many resort to alternative means of transport.)
3. The grade one gets in course is a function of the time and effort one invests in the course.

These are rather imprecise. In mathematics, when we say that one variable quantity is a function of another variable quantity we mean that their "values" are somehow related in some precise way. Here are two examples:

Example 1 *A child's dosage for a cough syrup is a function of the weight of the child. By this, we mean that every weight of a child, (to the nearest pound), may be matched with a precise dosage. The table below, pulled from a bottle of an off-the-counter children's cough syrup, gives such a matching for children weighing between 24 and 95 pounds*

Weight of child, to the nearest pound	*24-35*	*36-47*	*48-59*	*60-71*	*72-95*
Dosage in teaspoons	1	$1\frac{1}{2}$	*2*	$2\frac{1}{2}$	*3*

Example 2 *The letter grade one gets in most mathematics courses is a function of the numeric grade, which is a whole number between 0 and 100, one scores in the course. The table below shows how letter grades are typically related to the numeric grades.*

Number of point earned	*0 - 56*	*57 - 67*	*68 - 78*	*79 - 89*	*9 0 - 100*
Letter grade earned	*F*	*D*	*C*	*B*	*A*

In general, whenever one variable is a function of another variable in a precise way, we can construct a table with two rows, or two columns, showing precisely how they are related. For this reason, we define a function as follows:

Definition 3 *A function is a table with two rows, (or two columns), that shows how two variable quantities are related. It must have the property that every element in the first row (or the first column in the case of a table with two columns), is matched with exactly one element in the second row, (or the second column).*

The variable whose values are listed in the first row, (or the first column in the case of a table with two columns), is called the independent variable. The other variable is called the dependent variable. In the cough syrup example, the independent variable is the weight of the children and the dependent variable is the dosage.

Note that two or more elements in the first row of a function may be matched with the same element in the second row. In the cough syrup example, the twelve weights 36, 37, ... , 46, 47 in the first row are all matched with the same number $1\frac{1}{2}$ in the second row. What is unacceptable is to have an element in the first row that is matched with two or more elements in second row. For example, it would be absurd to prescribe 2 teaspoon to a 49-pound child and also prescribe 3 teaspoons to the same child.

Example 4 *The table below showing how the monthly USA Federal tax withholding of a single employee is related to the employee's monthly salary is a function.*

If Monthly Salary is			*The amount of income tax withheld monthly is*
Not over $175			*$0*
Over	*But not over*		
$175	$883	⋯	*$0.00 plus 10% of the amount the salary exceeds $175*
$883	$3,050	⋯	*$70.80 plus 15% of the amount the salary exceeds $883*
$3,050	$7,142	⋯	*$395.85 plus 25% of the amount the salary exceeds $3,050*
$7,142	$14,708	⋯	*$1,418.85 plus 28% of the amount the salary exceeds $7,142*
$14,708	$31,771	⋯	*$3,537.33 plus 33% o f the amount the salary exceeds $14,708*
$31,771	....	⋯	*$9,168.12 plus 35% of the amount the salary exceeds $31,771*

For example, consider a taxable monthly salary of $3550. It is over $3,050 but it is not over $7,142. To figure out the amount of income tax withheld monthly, go to the row "over $3050 but not over $7142". It states that the monthly income tax paid by an unmarried employee with a taxable income of $3550 per month is $395.85 plus 25% of (3550-3,050). This adds up to $395.85 + \frac{25}{100} \times 500 = 520.85$ *dollars*

What is the monthly tax paid by an unmarried employee with a taxable income of

(a) $1800 *(b) $7879* *(c) $150* *(d) $120880* *(e) $30,030*

What is the monthly taxable income of an unmarried employee whose monthly tax is

(a) $600 *(b) $35* *(c) $1500* *(d) $777* *(e) $20000*

Some New Terms

Given a function, (i.e. a two-row table that qualifies to be a function), the set of all the elements that may be put in the top row is called the ***domain*** *of the function. The* ***range*** *of the function consists of every element that may go in the bottom and can be matched with at least one element of the top row*

Example 5 *The domain of the cough syrup function is the set of all the integers from 24 to 95. Its range is the set of numbers* 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3.

Example 6 *The domain of the letter grade function is the set of integers from 0 to 100. Its range is the set of letters* A, B, C, D, F.

Example 7 *The domain of the tax function above is the set of all the non-negative numbers with at most two decimal places, (because salaries are quoted in dollars and cents, therefore we do not carry more than two decimal places). For the same reason, its range is a subset of the set of non-negative numbers with at most two decimal places.*

It is convenient to denote a function by a letter of the alphabet. The common choices are f, g, h among many. For example, we may denote the cough syrup function by the letter D, (for dosage), the letter grade function by g, (for grade) and the tax function by T, (for tax).

Say a given function, (i.e. a table with two rows satisfying the appropriate conditions) is denoted by a letter f. The element of the second row that is assigned to a given element x in the first row is called the **image** of x under f, or the **value** of f at x and it is denoted by $f(x)$.

Example 8 *Let D be the cough syrup function given above. Say we choose the number 40 in its domain. It is matched with the number $1\frac{1}{2}$ in the second row therefore the image of 40 under D is $1\frac{1}{2}$ and we may write*

$$D(40) = 1\tfrac{1}{2}$$

DO NOT mistake $D(40)$ for the product of D and 40; there is no multiplication involved. Think of $D(40)$ as an abbreviation for the phrase "the dosage for a 40 pound child". More examples:

$$D(31) = 1 \qquad D(27) = 1 \qquad D(88) = 3 \qquad D(70) = 2\tfrac{1}{2} \qquad D(91) = 3 \qquad D(60) = 1\tfrac{1}{2}$$

Example 9 *Let g be the letter grade function given above. A number like 85 is matched with the letter B, therefore we may write $g(85) = B$, which you should consider to be an abbreviation for the phrase "the grade assigned to 85 points is B", (NOT g multiplied by 85). The following are more images of g:*

$$g(70) = B \qquad g(88) = B \qquad g(52) = F \qquad g(65) = D \qquad g(92) = A \qquad g(68) = C$$

Example 10 *Let T be the tax function above. Then, as we showed above, $T(3550) = 520.85$. For another value, we choose a salary of \$8000. We find that $T(8000) = 1418.85 + \frac{28}{100} \times (8000 - 7142) = 1659.09$. Calculate the following:*

(a) $T(4545)$ *(b)* $T(7000)$ *(c)* $T(10,000)$ *(d)* $T(1,000,000)$ *(e)* $T(75)$

In the examples of the cough mixture and the grades we gave above, we were able to list all the numbers in the domain of each function. This is rarely the case. Here is one example in which we cannot:

Example 11 *A truck rental company charges \$40.00 per day plus mileage at the rate of 10 cents per mile. Suppose you rent a truck from this company for one day. The amount of money you will be charged is a function of the number of miles you clock. If, for example, you clock 125.6 miles then you will be charged*

$$40 + \tfrac{10}{100} \times 125.6 = 40 + 12.56 = 52.56 \text{ dollars}$$

It is clearly impossible to give a table listing all the possible mileages you may clock and the corresponding charges because they are infinitely many. One way around this problem is to use an algebraic expression to describe the possible pairs in the table. To this end, note that if you clock x miles (x can be any positive number) then you pay

$$40 + \tfrac{10}{100} \times x = 40 + 0.1x \text{ dollars}$$

In other words, an arbitrary mileage x corresponds to a charge of $40 + 0.1x$ dollars. Now, to display this function as a table, we list a few pairs with concrete mileages plus a pair with a general mileage x as shown below

Mileage	*10*	*59*	*125*	*200*	x
Charge in dollars	*41*	*45.90*	*52.50*	*60*	$40 + 0.1x$

Example 12 *This example introduces the* ***ceiling*** *function. It pairs a given real number x with the smallest integer that is bigger than or equal to x. There is a special symbol for the smallest integer that is bigger than or equal to x and it is $\lceil x \rceil$. For example, $\lceil 3.1 \rceil = 4$, $\lceil 3.9 \rceil = 4$. The following is a table of sample values for this function.*

Real number x	*6.1*	*8.9*	*12.01*	15.7	*0.2*	-2.5	-18.8	5	-7	π	x
Ceiling of x	*7*	9	13	*16*	1	-5	-18	*5*	-7	*4*	$\lceil x \rceil$

Example 13 *The "opposite" of the ceiling function is called the* ***floor*** *function. It pairs a given real number x with the largest integer that is smaller than or equal to x. There is a special symbol for the largest integer that is smaller than or equal to x and it is $\lfloor x \rfloor$. For example, $\lfloor 5.2 \rfloor = 4$, $\lfloor -3.1 \rfloor = -4$. Complete the following table of sample values for this function.*

Real number x	*6.5*	*9.9*	*12.4*	25.7	*0.2*	-2.5	-18.8	5	-7	π	x
floor of x											$\lfloor x \rfloor$

Formula for a Function

Denote the function in the truck rental example above, by C (for cost). We pointed out that the cost of x miles is $40 + 0.1x$ dollars, therefore we may write $C(x) = 40 + 0.1x$. Once again, think of $C(x)$ as the cost of renting the truck and drive it x miles. It is not C multiplied by x. In particular, $C(10) = 41$ and $C(200) = 60$. The equation $C(x) = 40 + 0.1x$ enables us to determine the image of any number x in the domain of C by simply substituting the value of x into the equation. For this reason, we call it the "**formula** for C".

In general, if there is an algebraic equation which we may use to determine the image of any element in the domain of a given function f then we call it a formula for f.

Because a table is so cumbersome to draw, whereas a formula is generally easy to write down, it is customary to define a function by giving its formula. We then agree that its domain is the set of all the numbers that can be substituted into the formula.

Example 14 *Let f be the function with formula $f(x) = 6x + 1$. As a table, (which you do not have to give), this function has the following sample pairs*

x	-1	0	2	2.1	x
$f(x)$	-5	1	13	7.6	$6x + 1$

Its domain is the set of all numbers (because any number x can be substituted into $6x + 1$).

Example 15 *Let g be the function with formula $g(x) = \frac{x+1}{x-2}$. As a table, this function has the following sample pairs.*

x	-1	0	1	$3,5$	x
$g(x)$	0	$-\frac{1}{2}$	-2	3	$\frac{x+1}{x-2}$

Note that with the exception of the number 2, all the numbers can be substituted into the expression $\frac{x+1}{x-2}$. Therefore the domain of this function is the set of all the numbers except 2. Using interval notation, the domain may be given as $(-\infty, 2) \cup (2, \infty)$.

Example 16 *Let h be the function with formula $h(x) = \sqrt{x+5}$. As a table, this function has the following sample pairs.*

x	-4	-3	0	3.1	x
$h(x)$	1	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{8.1}$	$\sqrt{x+5}$

All the numbers that are bigger than or equal to -5 may be substituted into the expression $\sqrt{x+5}$. The numbers smaller than -5 cannot be substituted into $\sqrt{x+5}$ because doing so requires us to find square roots of negative numbers. Therefore the domain of h is the set of all the numbers bigger than or equal to -5, denoted by $[-5, \infty)$.

Example 17 *Let w be the function with formula*

$$w(x) = \frac{\sqrt{4-x}}{x+2}.$$

Clearly, x cannot be -2 and it cannot be bigger than 4. Therefore the domain of W is the set $(-\infty, -2) \cup (-2, 4]$.

Exercise 18

1. *Construct a table like the one in Example 11, for a car rental company that charges \$25.00 per day plus 8 cents per mile.*

2. *A rectangle must have an area of 84 square inches. Complete the following table that shows how the width of the rectangle is related to its length:*

Length of rectangle	*3*	*4*	*21*		*30*	*40*	x
Width of rectangle	*28*			*18*			

Denote it by W. Determine its domain and formula.

3. *The cost of a telephone call between some two cities is calculated as follows: Any call lasting 2 or less minutes is charged a flat \$0.80. For a call lasting more than 2 minutes, the customer is charged the flat \$0.80 plus \$0.05 for each next minute beyond 2 minutes. (For example, all calls lasting over 7 but not more than 8 minutes are charged the flat \$0.80 plus* $\$0.05 \times 6$ *for the additional 6 minutes.) Complete the following table that shows how the cost of a call is related to how long it lasts:*

Length of call in minutes	*1.5*	*10*	*25.8*	*35*		*75.1*	*95*	$x > 2$
Cost of call in dollars		*1.20*			*2.00*			

Say we denote the function by C. Determine its domain and use the ceiling function of Example **??** *to write down a formula for C.*

4. *A Gym has two health plans; A and B. Under plan A, you pay a membership fee of \$40.00 and a monthly fee of \$25.00. Under plan B, you pay a membership fee of \$15.00 and a monthly fee of \$30.00. Complete the following tables:*

# of months as a Plan A member	*2*	*5*	*6*	*9*	*10*	x
Total amount paid						

# of months as a Plan B member	*2*	*5*	*6*	*9*	*10*	x
Total amount paid						

A prospective member would like to know how long it takes the total amount one pays under plan B to exceed the total amount one pays under plan A by more than \$87. How many months does it take? To get full credit, you have to show how you arrive at your answer.

5. *A function may not have a formula. The following is an example. It is a table in which the first row contains the positive integers bigger than 1 and every integer n in the first row is paired with the number of prime numbers less than n. Complete the part of the table given below.*

Integer $n > 1$	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*
Number of prime numbers less than n	*0*	*1*													