Sets and Set Notation

A number of practical problems require us to consider collections of things. For example, a survey of unemployment may require determining the collection of cities with high unemployment (say 15%). A study of birthrates may require us to consider countries with low birthrates (e.g. 5 births per 1000 people). Such collections are called sets. More precisely, a set is a collection of well-defined things or objects. The objects are dictated by the problem at hand. They may be numbers, letters, people, etc.

Some simple examples of sets:
A coin collection.
The collection of items in your room.
The collection of letters in the English alphabet.
The collection of all the real numbers
The collection of the even numbers.

The set of the letters in the English alphabet has 26 items. It is easy to think of a set with as many items as you like. For example, the set consisting of the mayor of your city has one item. The set of countries you have visited may contain a couple of items. Actually, a set can have an infinite number of items! The set of even numbers is an example.

Each item in a set is known as an **element** of the set. For example, p is an element of the set of the letters in the English alphabet and 12 is an element of the set of even numbers. Note that 7 is not an element of the set of even numbers and π is not an element of the set of letters in the English alphabet.

Sets may be written using a pair of braces $\{\quad\}$, with their elements listed in between.
For example the collection of the letters in the English alphabet could be written as

$$\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

and the collection of the even numbers may be written as

$$\{0, 2, -2, 4, -4, 6, -6, 8, -8, 10, -10, ...\}$$

The dots at the end are used to indicate that the pattern goes on indefinitely. Because we have listed the elements in the given sets, this method of writing a set is called the list method.

An alternative method of writing a given set is to write a phrase describing its elements and enclose it in a pair of braces $\{\quad\}$. As you would expect, this is called the descriptive method since it involves describing the elements of the set. For example, the collection of the letters of the English alphabet may be written as

$$\{\text{letters in the English alphabet}\}$$

and the collection of even numbers may be written as

$$\{\text{the even numbers}\}$$

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It is convenient to label a given set with a letter of the alphabet. One may also view this as giving the set a name. Capital letters are preferable as labels or names of sets. For example, we may label the set of letters in the English alphabet by A. Thus we write

$$A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

We may label the set of even numbers by E. Thus

$$E = \{0, 2, -2, 4, -4, 6, -6, 8, -8, 10, -10, ...\}$$

Some sets of numbers are denoted by special symbols:

1. The set of integers, (i.e., the negative whole numbers, zero and the positive whole numbers), is given the special symbol $\mathbb{Z}$.
2. The set of rational numbers, (these are the fractions like $\frac{3}{4}$, $\frac{19}{8}$, etc), is given the special symbol $\mathbb{Q}$.
3. The set of real numbers is given the special symbol $\mathbb{R}$.
4. The set of all the real numbers between two given real numbers a and b, (with $a < b$), is called an interval with endpoints a and b. It is denoted by a symbol that depends on whether or not the endpoints are included in the set.

   (a) If both end points are included in the set then it is called the closed interval with endpoints a and b and it is denoted by $[a, b]$. Using the notation introduced above, we may write it as

   $$[a, b] = \{\text{All the real numbers between } a \text{ and } b\text{, including } a \text{ and } b\}$$

   Alternatively,

   $$[a, b] = \{\text{All the real numbers } x \text{ such that } a \leq x \leq b\}$$

   An even fancier description of $[a, b]$ is

   $$[a, b] = \{\text{All the real numbers } x : a \leq x \leq b\}$$

   The full colon : is used to stand for the phrase "such that".

   Example 1 $[3, 8]$ *denotes the closed interval with end points 3 and 8. It consists of all the numbers that are bigger than or equal to 3 and are less than or equal to 8. A picture of the interval to help you visualize the set is shown below.*

   

   The dots at 3 and 8 are used to indicate that the numbers 3 and 8 are included in the set.

   (b) If both end points are excluded from the set then it is called an open interval with endpoints a and b and it is denoted by (a, b). Using the notation introduced above, we may write it as

   $$(a, b) = \{\text{All the real numbers between } a \text{ and } b\text{, excluding } a \text{ and } b\}$$

   Alternatively,

   $$(a, b) = \{\text{All the real numbers } x \text{ such that } a < x < b\}$$

   OR

   $$(a, b) = \{\text{All the real numbers } x : a < x < b\}$$

   Example 2 $(-4, 7)$ *denotes the open interval with endpoints -4 and 7. It consists of all the real numbers that are strictly bigger than -4 and are strictly less than 7. The figure below shows a picture of the interval.*

   

   The o's at -4 and 7 signify that the endpoints are excluded from the set.

(c) If one of the two endpoints is excluded from the set then it is called a half-open interval, (or a half-closed interval). It is denoted by $(a, b]$ if a is excluded but b is included and by $[a, b)$ if a is included but b is excluded.

Example 3 $(-1, 10]$ *denotes the half-open interval with endpoints* -1 *and* 10 *consisting of all the real numbers that are strictly bigger than* -1 *and are less than or equal to 10. A picture of the interval is shown below.*

Example 4 $[3.1, 9)$ *denotes the half-open interval with endpoint 3.1 and 9 consisting of all the real numbers that are bigger than or equal to 3.1 but are strictly smaller than 9. Draw a picture.*

5. The set of all the real numbers that are strictly bigger than a given real number a is called the open interval from a to infinity and it is denoted by (a, ∞). Thus a is excluded from the collection. If a is included then we get the set of all real numbers bigger than or equal to a, called the closed interval form a to infinity. It is denoted by $[a, \infty)$.

   Example 5 $(-12, \infty)$ *is the set of all the real numbers that are strictly bigger than* -12*. A picture of the interval is shown below.*

The dots are used to indicate that the line continues to the right indefinitely.

   Example 6 $[4, \infty)$ *is the set of all the real numbers that are bigger than or equal to 4. Draw a picture.*

6. The set of all the real numbers that are strictly less than a given real number b is called the open interval from minus infinity to b and it is denoted by $(-\infty, b)$. Thus b is excluded from the collection. If it is included then we get the closed interval from $-\infty$ to b, denoted by $(-\infty, b]$.

   Example 7 $(-\infty, 13.4)$ *is the set of all the real numbers that are strictly less than* 13.4*. Draw a picture of this interval.*

   Example 8 $(-\infty, -6]$ *is the set of all the real numbers that are less than or equal to* -6*.*

   Remark 9 *The symbols* $-\infty$*, (called minus infinity), and* ∞*, (called infinity), are not a real numbers, therefore expressions like* $[-\infty, 3)$ *or* $[4, \infty]$ *are not sets of real numbers.*

Some More Terms

To introduce more terms, consider the sets A and E, we introduced above, of the letters of the alphabet and the even numbers respectively. Clearly, p is an element of A, 12 is an element of E, π is not an element of A and 73 is not an element of E. A short way of writing that p is an element of A is

$$p \in A.$$

The symbol $\in$ stands for the phrase "is an element of". Using the symbol again, we may write

$$80 \in E.$$

The symbol $\notin$ is used to stand for the phrase "is not an element of". For example,

$$\pi \notin A \quad \text{and} \quad 73 \notin E.$$

These symbols enable us to write a number of sentences involving elements of sets more briefly.

More examples:
Let V be the set of vowels in the English alphabet and J be the set of integers from 1 to 10 inclusive. Then $e \in V$, $5 \in J$, $f \notin V$, $18 \notin J$.

Consider the table below which gives the unemployment and inflation rates for some country for the years from 1986 to 1999.

Year	1986	1987	1988	1989	1990	1991	1992
Unemployment rate(%)	6.6	6.1	5.4	5.2	5.4	6.6	7.4
Inflation rate (%)	1.9	3.6	4.1	4.8	5.4	4.2	3.0

Year	1993	1994	1995	1996	1997	1998	1999
Unemployment rate (%)	6.8	6.0	5.5	5.4	4.9	4.5	4.2
Inflation rate (%)	2.6	2.6	2.5	3.4	1.7	1.6	2.7

We now introduce the following sets:

$$\begin{aligned} A &= \{\text{years from 1986 to 1999 in which unemployment was at least 6\%}\} \\ B &= \{\text{years from 1986 to 1999 in which the inflation was at least 4\%}\} \\ C &= \{\text{years from 1986 to 1999 in which unemployment was below 5\%}\} \end{aligned}$$

Then

$$A = \{1986, 1987, 1991, 1992, 1993, 1994\},$$
$$B = \{1988, 1989, 1990, 1991\}$$
$$C = \{1997, 1998, 1999\}.$$

Clearly, $1986 \in A$, $1999 \in C$, $1994 \notin B$, $1996 \notin A$ and $1991 \in B$.

The Intersection of Given Sets
Consider the set $S = \{0, 3, 6, 9, 12, 15, 18\}$ of the first seven nonnegative multiples of 3 and the set $W = \{0, 2, 4, 6, 8, 10, 12, 14\}$ of the first eight nonnegative even numbers. The two sets have some elements in common, and they are 0, 6, and 12. The set consisting of the elements they share is called the intersection of the two sets. In this case we denote it by $S \cap W$. Thus

$$S \cap W = \{0, 6, 12\}$$

In general, the intersection of two given sets X and Y is the set, denoted by $X \cap Y$, consisting of all those elements which X and Y have in common.

The Union of Given Sets
Going back to the sets S and W above, the set we get by combining all the elements of S with the elements of W is called the union of S and W. It is denoted by $S \cup W$. Thus

$$S \cup W = \{0, 2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 18\}$$

Note that we **do not** write the elements 0,6,12 two times.

In general, the union of two given sets X and Y is the set, denoted by $X \cup Y$, consisting of all the elements that are in X or Y. Note that this includes the elements that are in both sets

.

Example 10 *Consider the sets A, B, and C above. Then $A \cap B$ is the set of years when the unemployment was at least 6% and the inflation rate was at least 4%. As the table shows, there was only one year and it was 1991. Therefore*

$$A \cap B = \{1991\}$$

On the other hand, $A \cup B$ is the set of the years when the unemployment was at least 6% or the inflation rate was at least 6%.

$$A \cup B = \{1986,\ 1987,\ 1988,\ 1989,\ 1990,\ 1991,\ 1992,\ 1993,\ 1994\}.$$

Example 11 *Let $A = [1, 7]$, $B = (-3, 5)$, $C = (-\infty, 2)$ and $D = [-1, \infty)$. Then*

(a) $A \cup B = (-3, 7]$ *(b)* $A \cap B = [1, 5)$ *(c)* $A \cap C = [1, 2)$ *(d)* $A \cap D = A$

(e) $B \cap C = (-3, 2)$ *(f)* $B \cup C = (-\infty, 5)$ *(g)* $A \cup D = D$ *(h)* $C \cap D = [-1, 2)$

The empty set

The set that contains no elements is called the empty set and it is denoted by $\varnothing$. There are many considerations which lead to the empty set. For example, if we take the sets A and C we obtained from the above table, we note that there is no year in which the inflation rate was at least 6% and the unemployment rate was below 5%. Therefore $A \cap C$ contains no element, so it must be empty and we may write

$$A \cap C = \varnothing$$

Subsets

There are many instances in which two sets V and W are given and each element of one of the sets, say V, is also an element of the other set. In such a case, we say that V is a subset of W. We use the symbol $\subset$ to stand for the phrase "is a subset of". Thus we write $V \subset W$ to mean that V is a subset of W.

Example 12 *Let E be the set of even numbers and S be the set of multiples of 6. Since every multiple of 6 is even, S is a subset of E and we may write $S \subset E$.*

Example 13 *Let V be the set of letters in the word GOOGLE and W be the set of letters in the name GOLDEN GATE. Since $V = \{E, G, L, O\}$ and $W = \{A, E, G, L, N, O, T\}$, it is clear that $V \subset W$.*

Example 14 *Let $\mathbb{Z}$ be the set of integers, E be the set of even numbers and O be the set of odd numbers. Then $E \subset \mathbb{Z}$ and $O \subset \mathbb{Z}$..*

A Universal Set

In all the problems we will consider, there is a single set which contains all the elements and the sets that are relevant to the problem. Such a set is called the universal set for the problem.

For example, the set of all numbers may be taken as the universal set in problems involving numbers, and the set of integers may be taken as the universal set in problems involving whole numbers.

In the problems to come, the universal set will either be given or it will be easy to identify.

Complement of a set

Assume you have been given a universal set U. Suppose X is a subset of the given universal set. Then the complement of X is the set of elements in U which are not in X. It is denoted by X^c. For example, let U be the set of the letters of the English alphabet and X be the set of vowels in the English alphabet. Then the complement of X is the set of consonants in the English alphabet. More precisely

$$\begin{aligned} U &= \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\} \\ X &= \{a, e, i, o, u\} \\ X^c &= \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\} \end{aligned}$$

Problem 15 *Let $A = [3, 8]$, $B = (-7, 5)$, $C = (-\infty, 12)$, $D = [2, \infty)$ and the universal set be the set U of all real numbers. Determine the following: (a) $B \cup D$ (b) $(C \cap D)$ (c) $(A \cap B)$ (d) $A \cup B$ (e) C^c, (the complement of C) (f) $(C \cap D)^c$ (g) $(A \cup B) \cap C$ (h) $(B \cap D) \cup A$*

Another example: Let $U = \{\text{all people}\}$, (this is the universal set for this example), $V = \{\text{people who like vanilla ice cream}\}$, and $C = \{\text{people who like chocolate ice cream}\}$. Then

$$\begin{array}{lcl} V^c & = & \{\text{people who do not like vanilla ice cream}\} \\ C^c & = & \{\text{people who do not like chocolate ice cream}\} \\ V \cap C^c & = & \{\text{people who like vanilla ice cream but do not like chocolate ice cream}\} \\ V \cup C & = & \{\text{people who like vanilla ice cream or chocolate ice cream}\} \end{array}$$

You will soon be asked to write a given set using set notation. Here are some examples:

The set $\{\text{people who like both vanilla and chocolate ice cream}\}$ may be written as $V \cap C$.
The set $\{\text{people who do not like any of the two flavors}\}$ is $(V \cup C)^c$
The set $\{\text{people who like only chocolate ice cream}\}$ is $C \cap V^c$